ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $a \geq b \geq c$ and $d, e, f \geq 0, e = \text{mid}\{d, e, f\}$,

the following relationship holds :

$$d.\left(r_a - \frac{w_b w_c}{w_a}\right) + e.\left(r_b - \frac{w_c w_a}{w_b}\right) + f.\left(r_c - \frac{w_a w_b}{w_c}\right) \geq 0$$

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$$r_a - \frac{w_b w_c}{w_a} = \frac{s\tan\frac{A}{2}.\frac{2bc}{b+c}.\cos\frac{A}{2} - \frac{2ca}{c+a}\cos\frac{B}{2}.\frac{2ab}{a+b}.\cos\frac{C}{2}}{w_a}$$

$$= \frac{2bc}{w_a}\left(\frac{s\sin\frac{A}{2}}{b+c} - \frac{2a^2.s}{4R(c+a)(a+b)\cos\frac{A}{2}}\right)$$

$$= \frac{2abcs}{w_a}\left(\frac{(c+a)(a+b) - 2a(b+c)}{(a+b)(b+c)(c+a).4R\cos\frac{A}{2}}\right) = \frac{2abcs(a-b)(a-c)}{(a+b)(b+c)(c+a).4R.\frac{2bc}{b+c}.\frac{s(s-a)}{bc}}$$

$$= \frac{2abcs(s+s-a)(s-b)(s-c)(x-y)(x-z)}{(a+b)(b+c)(c+a)(s-a)(s-b)(s-c).4R.2s}$$

$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \text{ and then : } a = y+z, b = z+x, c = x+y \\ \text{and } s = x+y+z \end{matrix}\right)$$

$$\therefore r_a - \frac{w_b w_c}{w_a} = \frac{abc(s+x)yz(x-y)(x-z)}{(a+b)(b+c)(c+a)(s-a)(s-b)(s-c).4R}.\text{and analogs}$$

$$\Rightarrow d.\left(r_a - \frac{w_b w_c}{w_a}\right) + e.\left(r_b - \frac{w_c w_a}{w_b}\right) + f.\left(r_c - \frac{w_a w_b}{w_c}\right)$$

$$= \frac{abc\big(d.(s+x)yz(x-y)(x-z) + e.(s+y)zx(y-z)(y-x) + f.(s+z)xy(z-x)(z-y)\big)}{(a+b)(b+c)(c+a)(s-a)(s-b)(s-c).4R}$$

$$= \frac{abc\left(\begin{matrix} s\left(\overbrace{d.yz(x-y)(x-z) + e.zx(y-z)(y-x) + f.xy(z-x)(z-y)}^{M}\right) + \\ xyz\left(\overbrace{d.(x-y)(x-z) + e.(y-z)(y-x) + f.(z-x)(z-y)}^{N}\right) \end{matrix}\right)}{(a+b)(b+c)(c+a)(s-a)(s-b)(s-c).4R}$$

$$= \text{LHS} \rightarrow (1)$$

$\boxed{\text{Case 1}}$ $d \geq e \geq f$ and then, we can assign : $e = f + m$ and $d = f + m + n$ (where $m, n \geq 0$) and then, following simplification :

$$M = nyz(x-y)(x-z) + mz^2(x-y)^2 + f\left(\sum_{\text{cyc}} x^2y^2 - xyz\left(\sum_{\text{cyc}} x\right)\right) \geq 0$$

$$(\because m, n, f \geq 0 \text{ and } \because a \geq b \geq c \Rightarrow z \geq y \geq x \Rightarrow (x-y)(x-z) \geq 0) \therefore M \geq 0 \rightarrow (2)$$

$$Also, \mathbf{N} = \mathbf{n}(x-\mathbf{y})(x-\mathbf{z}) + \mathbf{m}(x-\mathbf{y})^2 + \mathbf{f}\left(\sum_{cyc} x^2 - \sum_{cyc} x\mathbf{y}\right) \geq 0$$

$(\because \mathbf{m},\mathbf{n},\mathbf{f} \geq 0$ *and* $\because a \geq \mathbf{b} \geq \mathbf{c} \Rightarrow \mathbf{z} \geq \mathbf{y} \geq x \Rightarrow (x-\mathbf{y})(x-\mathbf{z}) \geq 0) \therefore \mathbf{N} \geq 0 \rightarrow (3)$

$$\therefore (1),(2),(3) \Rightarrow \mathbf{d}.\left(\mathbf{r}_a - \frac{\mathbf{w_b w_c}}{\mathbf{w}_a}\right) + \mathbf{e}.\left(\mathbf{r_b} - \frac{\mathbf{w_c w}_a}{\mathbf{w_b}}\right) + \mathbf{f}.\left(\mathbf{r_c} - \frac{\mathbf{w}_a \mathbf{w_b}}{\mathbf{w_c}}\right) \geq 0$$

Case 2 $\mathbf{f} \geq \mathbf{e} \geq \mathbf{d}$ *and* **then, we** *can assign* : $\mathbf{e} = \mathbf{d} + \mathbf{m}'$ *and* $\mathbf{f} = \mathbf{d} + \mathbf{m}' + \mathbf{n}'$
(**where** $\mathbf{m}',\mathbf{n}' \geq 0$) *and* **then, following simplification :**

$$\mathbf{M} = \mathbf{n}'\mathbf{yz}(x-\mathbf{y})(x-\mathbf{z}) + \mathbf{m}'\mathbf{z}^2(x-\mathbf{y})^2 + \mathbf{f}\left(\sum_{cyc} x^2\mathbf{y}^2 - x\mathbf{yz}\left(\sum_{cyc} x\right)\right) \geq 0$$

$$\left(\begin{matrix}\because \mathbf{m}',\mathbf{n}',\mathbf{f} \geq 0 \text{ and } \because a \geq \mathbf{b} \geq \mathbf{c} \Rightarrow \mathbf{z} \geq \mathbf{y} \geq x \Rightarrow \\ (x-\mathbf{y})(x-\mathbf{z}) \geq 0\end{matrix}\right) \therefore \mathbf{M} \geq 0 \rightarrow (4)$$

$$Also, \mathbf{N} = \mathbf{n}'(x-\mathbf{y})(x-\mathbf{z}) + \mathbf{m}'(x-\mathbf{y})^2 + \mathbf{f}\left(\sum_{cyc} x^2 - \sum_{cyc} x\mathbf{y}\right) \geq 0$$

$$\left(\begin{matrix}\because \mathbf{m}',\mathbf{n}',\mathbf{f} \geq 0 \text{ and } \because a \geq \mathbf{b} \geq \mathbf{c} \Rightarrow \mathbf{z} \geq \mathbf{y} \geq x \Rightarrow \\ (x-\mathbf{y})(x-\mathbf{z}) \geq 0\end{matrix}\right) \therefore \mathbf{N} \geq 0 \rightarrow (5) \therefore (1),(4),(5) \Rightarrow$$

$$\mathbf{d}.\left(\mathbf{r}_a - \frac{\mathbf{w_b w_c}}{\mathbf{w}_a}\right) + \mathbf{e}.\left(\mathbf{r_b} - \frac{\mathbf{w_c w}_a}{\mathbf{w_b}}\right) + \mathbf{f}.\left(\mathbf{r_c} - \frac{\mathbf{w}_a \mathbf{w_b}}{\mathbf{w_c}}\right) \geq 0 \text{ and combining}$$

$$\text{both cases}, \mathbf{d}.\left(\mathbf{r}_a - \frac{\mathbf{w_b w_c}}{\mathbf{w}_a}\right) + \mathbf{e}.\left(\mathbf{r_b} - \frac{\mathbf{w_c w}_a}{\mathbf{w_b}}\right) + \mathbf{f}.\left(\mathbf{r_c} - \frac{\mathbf{w}_a \mathbf{w_b}}{\mathbf{w_c}}\right) \geq 0$$

$\forall \Delta$ **ABC with** $a \geq \mathbf{b} \geq \mathbf{c}$ *and* $\mathbf{d},\mathbf{e},\mathbf{f} \geq 0 \,\big|\, \mathbf{e} = \mathrm{mid}\{\mathbf{d},\mathbf{e},\mathbf{f}\}$,
" = " iff Δ ABC is equilateral (QED)