ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $a \geq b \geq c$ and $d, e, f \geq 0$, $e = \text{mid}\{d, e, f\}$, the following relationship holds :

$$d.(r_a h_a - h_b h_c) + e.(r_b h_b - h_c h_a) + f.(r_c h_c - h_a h_b) \geq 0$$

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$$r_a h_a - h_b h_c = \frac{2r^2s^2}{a(s-a)} - \frac{4r^2s^2}{bc} = 2r^2s^2.\frac{\big(bc - 2a(s-a)\big)(s-b)(s-c)}{abc(s-a)(s-b)(s-c)}$$

$$= 2r^2s^2.\frac{(a^2 + bc - ab - ac)(s-b)(s-c)}{abc(s-a)(s-b)(s-c)} = 2r^2s^2.\frac{(a-b)(a-c)(s-b)(s-c)}{abc(s-a)(s-b)(s-c)}$$

$$\therefore r_a h_a - h_b h_c = 2r^2s^2.\frac{yz(x-y)(x-z)}{abc(s-a)(s-b)(s-c)} \text{ and analogs}$$

$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \text{ and then : } a = y+z, b = z+x, c = x+y \\ \text{and } s = x+y+z \end{matrix}\right) \Rightarrow \text{LHS}$$

$$\geq \frac{2r^2s^2}{abcr^2s}\left(\overbrace{d.yz(x-y)(x-z) + e.zx(y-z)(y-x) + f.xy(z-x)(z-y)}^{M}\right) \rightarrow (1)$$

$\boxed{\text{Case 1}}$ $d \geq e \geq f$ and then, we can assign : $e = f + m$ and $d = f + m + n$ (where $m, n \geq 0$) and then, following simplification :

$$\therefore (1), (2) \Rightarrow d.(r_a h_a - h_b h_c) + e.(r_b h_b - h_c h_a) + f.(r_c h_c - h_a h_b) \geq 0$$

$\boxed{\text{Case 2}}$ $f \geq e \geq d$ and then, we can assign : $e = d + m'$ and $f = d + m' + n'$ (where $m', n' \geq 0$) and then, following simplification :

$$M = n'yz(x-y)(x-z) + m'z^2(x-y)^2 + f\left(\sum_{\text{cyc}} x^2y^2 - xyz\left(\sum_{\text{cyc}} x\right)\right) \geq 0$$

$$\left(\begin{matrix} \because m', n', f \geq 0 \text{ and } \because a \geq b \geq c \Rightarrow z \geq y \geq x \Rightarrow \\ (x-y)(x-z) \geq 0 \end{matrix}\right) \therefore M \geq 0 \rightarrow (3)$$

$\therefore (1), (3) \Rightarrow d.(r_a h_a - h_b h_c) + e.(r_b h_b - h_c h_a) + f.(r_c h_c - h_a h_b) \geq 0$ and combining both cases, $d.(r_a h_a - h_b h_c) + e.(r_b h_b - h_c h_a) + f.(r_c h_c - h_a h_b) \geq 0$ $\forall \Delta$ ABC with $a \geq b \geq c$ and $d, e, f \geq 0 \,|\, e = \text{mid}\{d, e, f\}$,

'' = '' iff Δ ABC is equilateral (QED)