ROMANIAN MATHEMATICAL MAGAZINE

In $any\ \Delta$ ABC the $following$ $relationship$ $holds$:

$$\frac{2m_a}{r_b+r_c}+\frac{r_b+r_c}{2m_a}\le\sqrt{\frac{R+6r}{2r}}$$

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$$\frac{2m_a}{r_b+r_c}=x\ (say)\ and\ we\ are\ to\ prove:x+\frac{1}{x}\overset{?}{\le}\sqrt{\frac{R+6r}{2r}}=t\ (say)$$

$$\Leftrightarrow x^2-xt+1\overset{?}{\le}0\Leftrightarrow 2x\boxed{\underset{①}{\overset{?}{\le}}}t+\sqrt{t^2-4}\text{ AND }2x\boxed{\underset{②}{\overset{?}{\ge}}}t-\sqrt{t^2-4}$$

$$\left(\because t^2-4=\frac{R-2r}{2r}\overset{Euler}{\ge}0\right)$$

$$\text{Now, }(b+c)^2\overset{?}{\ge}32Rr.\cos^2\frac{A}{2}=8r(r_b+r_c)=8r^2s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$$

$$=8(s-a)(s-b)(s-c).\frac{a}{(s-b)(s-c)}=4a(b+c-a)$$

$$\Leftrightarrow(b+c)^2+4a^2-4a(b+c)\overset{?}{\ge}0\Leftrightarrow(b+c-2a)^2\overset{?}{\ge}0\to\text{true}$$

$$\therefore b+c\ge\sqrt{32Rr}.\cos\frac{A}{2}\Rightarrow m_a\overset{Lascu}{\ge}\frac{b+c}{2}.\cos\frac{A}{2}\ge\sqrt{8Rr}.\cos^2\frac{A}{2}$$

$$\Rightarrow\frac{2m_a}{r_b+r_c}\ge\frac{4.\sqrt{2Rr}.\cos^2\frac{A}{2}}{4R.\cos^2\frac{A}{2}}=\sqrt{\frac{2r}{R}}\Rightarrow 2x\ge 2.\sqrt{\frac{2r}{R}}\overset{?}{\ge}t-\sqrt{t^2-4}$$

$$=\sqrt{\frac{R+6r}{2r}}-\sqrt{\frac{R-2r}{2r}}\Leftrightarrow\frac{8r}{R}\overset{?}{\ge}\frac{R+6r}{2r}+\frac{R-2r}{2r}-2.\sqrt{\frac{R+6r}{2r}}.\sqrt{\frac{R-2r}{2r}}$$

$$\Leftrightarrow 2.\sqrt{\frac{R+6r}{2r}}.\sqrt{\frac{R-2r}{2r}}\overset{?}{\ge}\frac{R^2+2Rr-8r^2}{Rr}$$

$$\Leftrightarrow\frac{(R+6r)(R-2r)}{r^2}\overset{?}{\ge}\frac{(R^2+2Rr-8r^2)^2}{R^2r^2}$$

$$\left(\because R^2+2Rr-8r^2=R^2-4r^2+2r(R-2r)\overset{Euler}{\ge}0\right)\Leftrightarrow 32r^3(R-2r)\overset{?}{\ge}0$$

$$\rightarrow \text{true via Euler} \Rightarrow \boxed{② \text{ is true}}$$

$$\text{Again, } 2x \overset{\text{Dang Ngoc Minh}}{\leq} \sqrt{\frac{R+2r}{r}}$$

$$\left(\begin{array}{c}\text{Reference : Inequality in Triangle by Dang Ngoc Minh} - 352; \\ \text{published at www.ssmrmh.ro}\end{array}\right)$$

$$\overset{?}{\leq} t + \sqrt{t^2 - 4} = \sqrt{\frac{R+6r}{2r}} + \sqrt{\frac{R-2r}{2r}}$$

$$\Leftrightarrow \frac{R+2r}{r} \overset{?}{\leq} \frac{R+6r}{2r} + \frac{R-2r}{2r} + 2.\sqrt{\frac{R+6r}{2r}}.\sqrt{\frac{R-2r}{2r}} \Leftrightarrow 2.\sqrt{\frac{R+6r}{2r}}.\sqrt{\frac{R-2r}{2r}} \overset{?}{\geq} 0$$

$$\rightarrow \text{true via Euler} \Rightarrow \boxed{① \text{ is true}} \therefore \text{both ① and ② are true and so,}$$

$$\frac{2m_a}{r_b + r_c} + \frac{r_b + r_c}{2m_a} \leq \sqrt{\frac{R+6r}{2r}} \ \forall \Delta\, ABC, '' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$