ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds:

$$\frac{r_a}{h_a+h_b}+\frac{r_b}{h_c+h_b}+\frac{r_c}{h_a+h_c}\geq\frac{20R^2-Rr+3r^2}{18r(R+r)}$$

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$$\frac{r_a}{h_a+h_b}+\frac{r_b}{h_c+h_b}+\frac{r_c}{h_a+h_c}\leq\frac{r_a}{4}\left(\frac{1}{h_a}+\frac{1}{h_b}\right)+\frac{r_b}{4}\left(\frac{1}{h_c}+\frac{1}{h_b}\right)+\frac{r_c}{4}\left(\frac{1}{h_a}+\frac{1}{h_c}\right)=$$

$$=\frac{1}{4}\left(r_a\left(\frac{1}{r}-\frac{1}{h_c}\right)+r_b\left(\frac{1}{r}-\frac{1}{h_a}\right)+r_c\left(\frac{1}{r}-\frac{1}{h_b}\right)\right)=\frac{1}{4}\left(\frac{(r_a+r_b+r_c)}{r}-\left(\frac{r_a}{h_c}+\frac{r_b}{h_a}+\frac{r_c}{h_b}\right)\right)=$$

$$=\frac{1}{4r}(r_a+r_b+r_c)-\frac{1}{4}\left(\frac{r_a}{h_c}+\frac{r_b}{h_a}+\frac{r_c}{h_b}\right)=\frac{R}{r}+\frac{1}{4}-\frac{1}{4}\left(\frac{r_a}{h_c}+\frac{r_b}{h_a}+\frac{r_c}{h_b}\right)$$

$$\frac{r_a}{h_c}+\frac{r_b}{h_a}+\frac{r_c}{h_b}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}=3\sqrt[3]{\frac{R}{2r}}$$

$$\frac{r_a}{h_a+h_b}+\frac{r_b}{h_c+h_b}+\frac{r_c}{h_a+h_c}\leq\frac{R}{r}+\frac{1}{4}-\frac{3}{4}\sqrt[3]{\frac{R}{2r}}$$

Let's prove that :

$$\frac{R}{r}+\frac{1}{4}-\frac{3}{4}\sqrt[3]{\frac{R}{2r}}\leq\frac{20R^2-Rr+3r^2}{18r(R+r)}$$

$$Let:\ \sqrt[3]{\frac{R}{2r}}=x\geq 1\ \ (R\geq 2r)$$

$$Then:\ 2x^3-\frac{1}{4}-\frac{3}{4}x\leq\frac{80x^6-2x^3+3}{18(2x^3+1)}\Leftrightarrow\frac{8x^3-3x+1}{4}\leq\frac{80x^6-2x^3+3}{18(2x^3+1)}$$

$$(8x^3-3x+1)(18x^3+9)\leq 160x^6-4x^3+6$$

$$16x^6+54x^4-94x^3+27x-3\geq 0$$

$$(x-1)\left(16x^5+16x^4+70x^3-24x^2-24x+3\right)\geq 0\ \ -to\ prove$$

$$(x-1)\geq 0 \;\rightarrow\; x\geq 1 \rightarrow \sqrt[3]{\frac{R}{2r}}\geq 1 \rightarrow\; R\geq 2r \;\; (True)$$

Remains to prove: $16x^5+16x^4+70x^3-24x^2-24x+3>0$

$$16x^5+16x^4+24x^2(x-1)+24x(x^2-1)+22x^3+3>0 \;\; (True)$$

$$Equality\;\; holds\;\; for: \quad a=b=c$$