ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{1}{12}\sum_{cyc} m_a(h_a+h_b)(h_a+h_c) \le \frac{1}{8}\prod_{cyc}(r_a+h_a) \le m_a m_b m_c$$

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$$h_a+h_b = 2F\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{2F(a+b)}{ab} \ \& \ h_a+h_c = \frac{2F(a+c)}{ac}$$

$$\frac{1}{12}\sum_{cyc} m_a(h_a+h_b)(h_a+h_c) = \frac{F^2}{3}\sum \frac{m_a(a+b)(a+c)}{a^2bc}$$

$$r_a+h_a = \frac{F}{s-a}+\frac{2F}{a} = \frac{F(2s-a)}{a(s-a)} = \frac{F(b+c)}{a(s-a)} \ \& \ F^2 = s(s-a)(s-b)(s-c)$$

$$\frac{1}{8}\prod_{cyc}(r_a+h_a) = \frac{F^3}{8}\frac{(a+b)(b+c)(c+a)}{abc(s-a)(s-b)(s-c)} = \frac{Fs(a+b)(b+c)(c+a)}{8abc}$$

$$\textit{we need to show: } \frac{1}{12}\sum_{cyc} m_a(h_a+h_b)(h_a+h_c) \le \frac{1}{8}\prod_{cyc}(r_a+h_a)$$

$$or, \frac{F^2}{3}\sum \frac{m_a(a+b)(a+c)}{a^2bc} \le \frac{Fs(a+b)(b+c)(c+a)}{8abc}$$

$$or, 8F\sum \frac{m_a(a+b)(a+c)}{a} \le 3s(a+b)(b+c)(c+a)$$

$$8F\sum \frac{m_a(a+b)(b+c)(a+c)}{a(b+c)} \le 3s(a+b)(b+c)(c+a)$$

$$\sum \frac{m_a}{a(b+c)} \le \frac{3s}{8F} = \frac{3}{8r}$$

$$\left(\sum \frac{m_a}{a(b+c)}\right)^2 \overset{C-S}{\le} \left(\sum m_a^2\right)\left(\sum \frac{1}{a^2(b+c)^2}\right) \overset{AM-GM}{\le} \left(\frac{3}{4}\sum a^2\right)\left(\sum \frac{1}{a^2 4bc}\right) \le$$

$$\overset{Leibniz}{\le} \left(\frac{3}{4}.9R^2\right)\left(\frac{1}{4}\sum \frac{bc}{a^2b^2c^2}\right) \le \frac{27}{16a^2b^2c^2}\frac{(a+b+c)^2}{3} = \frac{27\times 4s^2}{48\times 16R^2r^2s^2} = \frac{9}{64r^2}$$

$$then \sum \frac{m_a}{a(b+c)} \le \frac{3}{8r} \;\&\; \frac{1}{12}\sum_{cyc} m_a(h_a+h_b)(h_a+h_c) \le \frac{1}{8}\prod_{cyc}(r_a+h_a) \;\; (A)$$

$$\frac{1}{8}\prod_{cyc}(r_a+h_a) = \frac{F^3}{8}\frac{(a+b)(b+c)(c+a)}{abc(s-a)(s-b)(s-c)} = \frac{Fs(a+b)(b+c)(c+a)}{8abc}$$

$$= \frac{Fs(a+b)(b+c)(c+a)}{8\times 4RF} = \frac{s(a+b)(b+c)(c+a)}{32R}$$

$$m_am_bm_c \ge \frac{b+c}{2}\cos\frac{A}{2}\times\frac{a+c}{2}\cos\frac{B}{2}\times\frac{b+a}{2}\cos\frac{C}{2} = \frac{(a+b)(b+c)(c+a)}{8}\prod\cos\left(\frac{A}{2}\right) =$$

$$= \frac{(a+b)(b+c)(c+a)}{8}\frac{s}{4R} = \frac{s(a+b)(b+c)(c+a)}{32R}$$

$$\frac{1}{8}\prod_{cyc}(r_a+h_a) \le m_am_bm_c \;\; (B)$$

$$From\ (A)\ \&\ (B)\ we\ get:$$

$$\frac{1}{12}\sum_{cyc} m_a(h_a+h_b)(h_a+h_c) \le \frac{1}{8}\prod_{cyc}(r_a+h_a) \le m_am_bm_c$$

Equality holds for an equilateral triangle.