ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sin\frac{A}{2}.m_a+\sin\frac{B}{2}.m_b+\sin\frac{B}{2}.m_b\le\frac{\sqrt{3}}{2}\cdot s$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\sin\frac{A}{2}.m_a\right)=\sum_{cyc}\frac{\sqrt{a(s-b)(s-c)}.m_a}{\sqrt{4Rrs}}\overset{CBS}{\le}$$

$$\le\frac{1}{\sqrt{4Rrs}}.\sqrt{\sum_{cyc}((s-b)(s-c))}.\sqrt{\frac{1}{4}\sum_{cyc}\left(a\left(2\sum_{cyc}a^2-3a^2\right)\right)}=$$

$$=\frac{\sqrt{4Rr+r^2}}{\sqrt{4Rrs}}.\sqrt{2s(s^2-4Rr-r^2)-\frac{3}{4}.2s(s^2-6Rr-3r^2)}\overset{?}{\le}\frac{\sqrt{3}}{2}.s$$

$$\overset{squaring}{\Leftrightarrow}\frac{(s^2+2Rr+5r^2)(4R+r)}{8R}\overset{?}{\le}\frac{3s^2}{4}\Leftrightarrow(2R-r)s^2\underbrace{\overset{?}{\ge}}_{(*)}r(4R+r)(2R+5r)$$

$$\text{Now},(2R-r)s^2\overset{Gerretsen}{\ge}(2R-r)(16Rr-5r^2)\overset{?}{\ge}r(4R+r)(2R+5r)$$

$$\Leftrightarrow 24Rr(R-2r)\overset{?}{\ge}0\to\text{true via Euler}\Rightarrow(*)\text{ is true}$$

$$\therefore\sin\frac{A}{2}.m_a+\sin\frac{B}{2}.m_b+\sin\frac{B}{2}.m_b\le\frac{\sqrt{3}}{2}.s\ \forall\ ABC,$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$