ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{1}{r_a}+\frac{1}{m_a}\le\frac{4}{h_b+h_c}$$

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$$Let\ x=s-a, y=s-b, z=s-c\ then:$$

$$a=y+z, b=z+x, c=x+y\ \&\ s=x+y+z$$

$$F^2=s(s-a)(s-b)(s-c)=sxyz,\quad \frac{1}{r_a}=\frac{s-a}{F}=\frac{x}{F}\ (1)$$

$$m_a^2=\frac{2b^2+2c^2-a^2}{4}$$

$$m_a^2=\frac{2b^2+2c^2-a^2}{4}=\frac{4x^2+4xy+4xz+(y-z)^2}{4}=\frac{4xs+(y-z)^2}{4}\ \&$$

$$\frac{4}{h_b+h_c}=\frac{2bc}{F(b+c)}$$

$$We\ need\ to\ show:$$

$$\frac{1}{r_a}+\frac{1}{m_a}\le\frac{4}{h_b+h_c}\quad or$$

$$\frac{F}{r_a}+\frac{F}{m_a}\le\frac{4F}{h_b+h_c}$$

$$x+\frac{F}{m_a}\overset{(1)}{\le}\frac{2(x+y)(x+z)}{2x+y+z}\ or, \frac{F}{m_a}\le\frac{2(x+y)(x+z)}{2x+y+z}-x$$

$$\frac{F}{m_a}\le\frac{x(y+z)+2yz}{2x+y+z}\ or\ \frac{F^2}{m_a^2}\le\left(\frac{x(y+z)+2yz}{2x+y+z}\right)^2$$

$$\frac{4sxyz}{4xs+(y-z)^2}\le\left(\frac{x(y+z)+2yz}{2x+y+z}\right)^2$$

$$(x(y+z)+2yz)^2(4xs+(y-z)^2)-4sxyz(2x+y+z)^2 \geq 0$$

$$(xy+xz+2yz)^2(4x^2+4xy+4xz+y^2 - 2yz+z^2)-4xyz(x+y+z)(2x+y+z)^2 \geq 0$$

$$(\text{putting the value of } s=x+y+z)$$

$$4x^4y^2+4x^4z^2+8x^4yz+4x^3y^3+4x^3z^3-4x^4yz-4x^3y^2z-4x^3yz^2+x^2y^4+$$

$$+x^2z^4+8x^2y^3z+8x^2yz^3+10x^2y^2z^2+4y^4z^2-4y^3z^3+4y^2z^4 \geq 0$$

$$or, (y-z)^2(4x^4+4x^3(y+z)+x^2(y-z)^2+4y^2z^2) \geq 0 \text{ true}$$

$$\text{Equality holds for } y=z \text{ or } b=c$$