ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$r_b + r_c \geq max\left\{\frac{m_b}{m_c}r_b + \frac{m_c}{m_b}r_c; \frac{w_b}{w_c}r_b + \frac{w_c}{w_b}r_c; \frac{h_b}{h_c}r_b + \frac{h_c}{h_b}r_c\right\}$$

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Let $J \in \{m, w, h\}$ *represent the median, angle bisector or the altitude.*

It suffices to prove that:

$$r_b + r_c \geq \frac{J_b}{J_c}r_b + \frac{J_c}{J_b}r_c$$

WLOG, assume $b \geq c$.

$$Also\ let\ (a, b, c) = (y + z, z + x, x + y)\ so\ that\ z + x \geq x + y\ \therefore z \geq y$$

$$We\ know, b \geq c \Rightarrow J_b \leq J_c \Leftrightarrow \frac{1}{J_b} \geq \frac{1}{J_c}.$$

Lemma: $r_b J_b \geq r_c J_c$

Proof of the lemma:

1.) *For* h, *we have to prove that*: $r_b h_b \geq r_c h_c \Leftrightarrow$

$$\frac{\Delta}{s-b} \cdot \frac{2\Delta}{b} \geq \frac{\Delta}{s-c} \cdot \frac{2\Delta}{c} \Leftrightarrow \frac{1}{y} \cdot \frac{1}{z+x} \geq \frac{1}{z} \cdot \frac{1}{x+y} \Leftrightarrow$$

$$\Leftrightarrow zx + yz \geq yz + xy \Leftrightarrow zx \geq xy\ \ which\ is\ true\ since\ z \geq y.$$

2.) *For* w, *we have to prove that*: $r_b w_b \geq r_c w_c \Leftrightarrow$

$$\frac{\Delta}{s-b} \cdot \frac{2}{a+c}\sqrt{acs(s-b)} \geq \frac{\Delta}{s-c} \cdot \frac{2}{a+b}\sqrt{abs(s-c)} \overset{Squaring\ and\ simplifying}{\Longleftrightarrow}$$

$$\frac{c}{s-b} \cdot \frac{1}{(a+c)^2} \geq \frac{b}{s-c} \cdot \frac{1}{(a+b)^2}$$

$$\Leftrightarrow \frac{c(s-c)}{(a+c)^2} \geq \frac{b(s-b)}{(a+b)^2} \Leftrightarrow \frac{xz+yz}{(x+2y+z)^2} \geq \frac{xy+yz}{(x+y+2z)^2}\ \ which\ is\ true\ since\ z \geq y.$$

3.) *For* m, *we have to prove that*: $r_b m_b \geq r_c m_c \Leftrightarrow$

$$\frac{\Delta^2}{(s-b)^2}\left(\frac{2a^2 + 2c^2 - b^2}{4}\right) \geq \frac{\Delta^2}{(s-c)^2}\left(\frac{2a^2 + 2b^2 - c^2}{4}\right)$$

$$\Leftrightarrow \frac{2(y+z)^2 + 2(x+y)^2 - (z+x)^2}{y^2} \geq \frac{2(y+z)^2 + 2(z+x)^2 - (x+y)^2}{z^2}$$

which is equivalent to:

$$\frac{(z-y)}{y^2 z^2}\left[(y+z)\left(x - \frac{y^2 - yz + z^2}{y+z}\right)^2 + \frac{yz(8y^2 + 7yz + 8z^2)}{y+z}\right] \geq 0$$

$$which\ is\ also\ true\ since\ z \geq y.$$

$$Thus, the\ lemma\ is\ now\ proven!$$

We can notice that $\left(\frac{1}{J_b}, \frac{1}{J_c}\right)$ *and* $(r_b J_b, r_c J_c)$ *are similarly sorted.*

Hence, by Rearrangement Inequality we have:

$$\frac{1}{J_b}(r_bJ_b)+\frac{1}{J_c}(r_cJ_c)\geq\frac{1}{J_c}(r_bJ_b)+\frac{1}{J_b}(r_cJ_c)$$

$$\Leftrightarrow r_b+r_c\geq\frac{J_b}{J_c}r_b+\frac{J_c}{J_b}r_c$$

Therefore, $$r_b+r_c\geq max\left\{\frac{m_b}{m_c}r_b+\frac{m_c}{m_b}r_c;\frac{w_b}{w_c}r_b+\frac{w_c}{w_b}r_c;\frac{h_b}{h_c}r_b+\frac{h_c}{h_b}r_c\right\}$$

Equality holds if the triangle is isosceles $(b = c)$