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In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{r_a}{w_a}\geq\sum_{cyc}\frac{r_b+r_c}{h_b+h_c}\geq\sum_{cyc}\frac{m_a}{h_a}$$

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$$\sum_{cyc}\frac{r_b+r_c}{h_b+h_c}=\frac{4R}{2rs}\cdot\sum_{cyc}\left(\frac{s(s-a)}{bc}\cdot\frac{bc}{b+c}\right)$$

$$=\frac{2R}{r.2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left((s-a)\left(a^2+\sum_{cyc}ab\right)\right)$$

$$=\frac{R\left(2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)+s(s^2+4Rr+r^2)\right)}{r.s(s^2+2Rr+r^2)}$$

$$=\frac{R(s^2+8Rr+5r^2)}{r(s^2+2Rr+r^2)}\overset{(i)}{=}\sum_{cyc}\frac{r_b+r_c}{h_b+h_c}$$

$$\text{Now, }\left(\sum_{cyc}\frac{m_a}{h_a}\right)^2=\frac{1}{4r^2s^2}\left(\sum_{cyc}\left(\frac{a^2}{4}\left(2\sum_{cyc}a^2-3a^2\right)\right)+\sum_{cyc}\left(bc.\frac{2a^2+bc}{2}\right)\right)$$

(Reference : Solution to Inequality in Triangle – 316 by Dang Ngoc Minh; publshed at www.ssmrmh.ro)

$$=\frac{1}{4r^2s^2}.(2(s^2-4Rr-r^2)^2-(s^2+4Rr+r^2)+16Rrs^2+12r^2s^2+8Rrs^2)$$

$$\overset{?}{\leq}\left(\sum_{cyc}\frac{r_b+r_c}{h_b+h_c}\right)^2\overset{\text{via (i)}}{=}\frac{R^2(s^2+8Rr+5r^2)^2}{r^2(s^2+2Rr+r^2)^2}\Leftrightarrow$$

$$-s^8+(4R^2-4Rr-8r^2)s^6+r(64R^3+20R^2r-36Rr^2-14r^3)s^4+$$
$$r^2\overbrace{(256R^4+256R^3r+12R^2r^2-44Rr^3-8r^4)}^{M}s^2-$$
$$r^4\overbrace{(64R^4+96R^3r+52R^2r^2+12Rr^3+r^4)}^{N}\boxed{\overset{?}{\geq}\,①}\,0$$

Now, Double – Rouche $\Rightarrow$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①}\overset{?}{\geq}-s^4\overbrace{(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)}^{T}$$

$$\Leftrightarrow-(24R+6r)s^6+(128R^3+68R^2r-24Rr^2-13r^3)s^4+rMs^2-r^3N\boxed{\overset{?}{\geq}\,②}\,0$$

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Again $\because$ $P = -(24R+6r)T \geq 0$ $\therefore$ to prove ②, it suffices to prove : LHS of ② $\overset{?}{\geq} P$

$$\Leftrightarrow \overbrace{(32R^3 - 436R^2r - 96Rr^2 - r^3)s^4 + r(1792R^4 + 1792R^3r + 588R^2r^2 + 52Rr^3 - 2r^4)s^2}^{U} - r^3N \boxed{\overset{?}{\underset{(3)}{\geq}}} 0 \text{ and } \because U \overset{\text{Gerretsen}}{\geq}$$

$$U(16Rr - 5r^2) > r^3N \left(\because R \overset{\text{Euler}}{\geq} 2r\right) \therefore \text{if } 32R^3 - 436R^2r - 96Rr^2 - r^3 \geq 0,$$

then ③ is definitely true and if : $32R^3 - 436R^2r - 96Rr^2 - r^3 < 0$, then, since :

$$Q = (32R^3 - 436R^2r - 96Rr^2 - r^3)T \overset{\text{via (ii)}}{\geq} 0 \therefore \text{to prove ③, it suffices to prove :}$$

$$\text{LHS of ③} \overset{?}{\geq} Q \Leftrightarrow (8R^4 + 43R^3r - 461R^2r^2 - 29Rr^3 + 14r^4)s^2 \boxed{\overset{?}{\underset{(4)}{\geq}}}$$

$$r(128R^5 - 1648R^4r - 1664R^3r^2 - 611R^2r^3 - 99Rr^4 - 6r^5)$$

$\boxed{\textit{Case 1}}$ $X = 8R^4 + 43R^3r - 461R^2r^2 - 29Rr^3 + 14r^4 \geq 0$ and then :

$$\text{LHS of ④} \overset{\text{Gerretsen}}{\geq} X(16Rr - 5r^2) \overset{?}{\geq} \text{RHS of ④}$$

$$\Leftrightarrow 32t^6 + 76t^5 - 167t^3 - 832t^2 + 68t + 48 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow$$

$$(t-2)(32t^5 + 140t^4 + 280t^3 + 393t^2 - 46t - 24) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$\boxed{\textit{Case 2}}$ $X < 0$ and then : LHS of ④ $\overset{\text{Gerretsen}}{\geq} X(4R^2 + 4Rr + 3r^2) \overset{?}{\geq}$ RHS of ④

$$\Leftrightarrow 2296t^4 - 5927t^3 + 2452t^2 + 468t - 64 \overset{?}{\geq} 0$$

$\Leftrightarrow (t-2)(2296t^3 - 1335t^2 - 218t + 32) \overset{?}{\geq} 0 \to$ true $\because t \geq 2 \Rightarrow$ ④ is true; again,

$$\sum_{\text{cyc}} \frac{r_a}{w_a} \geq \frac{29R^2 - 2s^2 + 10Rr + 14r^2}{16Rr} \left(\begin{array}{c}\text{Inequality in Triangle} - 228 \text{ by Dang} \\ \text{Ngoc Minh; publshed at www.ssmrmh.ro}\end{array}\right)$$

$$\overset{?}{\geq} \frac{R(s^2 + 8Rr + 5r^2)}{r(s^2 + 2Rr + r^2)} \Leftrightarrow -2s^4 + (13R^2 + 6Rr + 12r^2)s^2 -$$

$$r(70R^3 + 31R^2r - 38Rr^2 - 14r^3) \boxed{\overset{?}{\underset{(5)}{\geq}}} 0 \text{ and it suffices to prove : LHS of ⑤} \overset{?}{\geq} -2T$$

$$\Leftrightarrow (5R^2 - 34Rr + 16r^2)s^2 + r(58R^3 + 65R^2r + 62Rr^2 + 16r^3) \boxed{\overset{?}{\underset{(6)}{\geq}}} 0 \text{ and it's}$$

trivially true when : $5R^2 - 34Rr + 16r^2 \geq 0$ and when : $5R^2 - 34Rr + 16r^2 < 0$,

$$\text{then : LHS of ⑥} \overset{\text{Gerretsen}}{\geq} (5R^2 - 34Rr + 16r^2)(4R^2 + 4Rr + 3r^2) +$$

$$r(58R^3 + 65R^2r + 62Rr^2 + 16r^3) \overset{?}{\geq} 0 \Leftrightarrow 10t^4 - 29t^3 + 4t^2 + 12t + 32 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)^2(10t^2 + 11t + 8) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ⑥ \Rightarrow ⑤ \text{ is true; so}$$

$$\sum_{\text{cyc}} \frac{r_a}{w_a} \geq \sum_{\text{cyc}} \frac{r_b + r_c}{h_b + h_c} \geq \sum_{\text{cyc}} \frac{m_a}{h_a} \ \forall \ \text{ABC}, '' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$