ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$r_b + r_c \ge max\left\{\frac{m_b}{m_c}.r_b + \frac{m_c}{m_b}.r_c; \frac{w_b}{w_c}.r_b + \frac{w_c}{w_b}.r_c; \frac{h_b}{h_c}.r_b + \frac{h_c}{h_b}.r_c\right\}$$

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$$r_b + r_c \overset{?}{\ge} \frac{m_b}{m_c}.r_b + \frac{m_c}{m_b}.r_c \overset{squaring}{\Leftrightarrow} r_b^2\left(1 - \frac{m_b^2}{m_c^2}\right) \overset{?}{\ge} r_c^2\left(\frac{m_c^2}{m_b^2} - 1\right)$$

$$\Leftrightarrow \frac{(m_c^2 - m_b^2)(r_b^2 m_b^2 - r_c^2 m_c^2)}{m_b^2 m_c^2} \overset{?}{\ge} 0$$

WLOG we may assume : $b \ge c$ and then : $m_c \ge m_b$ and so, it remains to prove :

$$(s-c)^2\left(2\sum_{cyc} a^2 - 3b^2\right) \overset{?}{\ge} (s-b)^2\left(2\sum_{cyc} a^2 - 3c^2\right)$$

$$\Leftrightarrow \left(2\sum_{cyc} a^2\right)(b-c)a \overset{?}{\ge} 3\big(b(s-c) + c(s-b)\big).s(b-c) \text{ and } \because b \ge c$$

$\therefore$ it remains to prove :

$$2((y+z)^2 + (z+x)^2 + (x+y)^2)(y+z) \overset{?}{\ge} 3(x+y+z)\big(z(z+x) + y(x+y)\big)$$

$$\left(\begin{matrix}\text{where } x = s-a, y = s-b, z = s-c \text{ and then :}\\ a = y+z, b = z+x, c = x+y \text{ and } s = x+y+z\end{matrix}\right)$$

$$\Leftrightarrow 3xy(x+y) + 2x(y+z)^2 + 2xyz + 3(y^3+z^3) + 7yz(y+z) \overset{?}{\ge} 0$$

$$\to \text{true} \because x, y, z > 0 \text{ (strict inequality)} \therefore \boxed{r_b + r_c \ge \frac{m_b}{m_c}.r_b + \frac{m_c}{m_b}.r_c}$$

$$\text{Again, } r_b + r_c \overset{?}{\ge} \frac{w_b}{w_c}.r_b + \frac{w_c}{w_b}.r_c \overset{squaring}{\Leftrightarrow} r_b^2\left(1 - \frac{w_b^2}{w_c^2}\right) \overset{?}{\ge} r_c^2\left(\frac{w_c^2}{w_b^2} - 1\right)$$

$$\Leftrightarrow \frac{(w_c^2 - w_b^2)(r_b^2 w_b^2 - r_c^2 w_c^2)}{w_b^2 w_c^2} \overset{?}{\ge} 0$$

WLOG we may assume : $b \ge c$ and then : $w_c \ge w_b$ and so, it remains to prove :

$$\frac{s}{s-b}.\frac{4ca}{(c+a)^2} \overset{?}{\ge} \frac{s}{s-c}.\frac{4ab}{(a+b)^2} \text{ and } \because (a+b)^2 \ge (c+a)^2$$

$$\therefore \text{ it suffices to prove : } c(s-c) \overset{?}{\ge} b(s-b) \Leftrightarrow (b+c)(b-c) \overset{?}{\ge} s(b-c)$$

$$\Leftrightarrow (2s-a-s)(b-c) \overset{?}{\ge} 0 \Leftrightarrow (s-a)(b-c) \overset{?}{\ge} 0 \to \text{true}$$

$$\therefore \boxed{r_b + r_c \ge \frac{w_b}{w_c}.r_b + \frac{w_c}{w_b}.r_c} \text{ and finally, } r_b + r_c \overset{?}{\ge} \frac{h_b}{h_c}.r_b + \frac{h_c}{h_b}.r_c$$

$$\overset{squaring}{\Leftrightarrow} r_b^2\left(1 - \frac{h_b^2}{h_c^2}\right) \overset{?}{\ge} r_c^2\left(\frac{h_c^2}{h_b^2} - 1\right) \Leftrightarrow \frac{(h_c^2 - h_b^2)(r_b^2 h_b^2 - r_c^2 h_c^2)}{h_b^2 h_c^2} \overset{?}{\ge} 0$$

WLOG we may assume : $b \ge c$ and then : $h_c \ge h_b$ and so, it remains to prove :

$$\mathbf{r_b h_b \overset{?}{\geq} r_c h_c \Leftrightarrow \frac{r.s^2}{2R}.sec^2\frac{B}{2} \overset{?}{\geq} \frac{r.s^2}{2R}.sec^2\frac{C}{2} \rightarrow true \because f(x) = sec\frac{x}{2}\ is \uparrow on\ (0,\pi)\ and}$$

$$\mathbf{\because b \geq c \therefore \boxed{r_b + r_c \geq \frac{h_b}{h_c}.r_b + \frac{h_c}{h_b}.r_c}\ and\ so, from\ the\ boxed\ relations,}$$

$$\mathbf{it's\ clear\ that: r_b + r_c \geq max\left\{\frac{m_b}{m_c}.r_b + \frac{m_c}{m_b}.r_c; \frac{w_b}{w_c}.r_b + \frac{w_c}{w_b}.r_c; \frac{h_b}{h_c}.r_b + \frac{h_c}{h_b}.r_c\right\}}$$

$$\mathbf{\forall\ ABC, '' = ''\ iff\ b = c\ (QED)}$$