ROMANIAN MATHEMATICAL MAGAZINE

In Δ ABC holds :

$$\frac{3\sum_{cyc}(23a^4-14b^2c^2)}{16\sum_{cyc}a^2} \geq \left(\sum_{cyc}\frac{m_a^2}{m_b+m_c}\right)^2 \geq \frac{3\sum_{cyc}(8a^4+b^2c^2)}{16\sum_{cyc}a^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{b+c} = \frac{(\sum_{cyc}a^2)(\sum_{cyc}ab)+2\sum_{cyc}a^2b^2-16r^2s^2}{2s(s^2+2Rr+r^2)}$$

$$\Rightarrow\left(4\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{a^2}{b+c}\right)^2$$

$$= \frac{8(s^2+4Rr+r^2)(s^4-(4Rr+r^2)^2+(s^2+4Rr+r^2)^2-16Rrs^2-8r^2s^2)^2}{s^2(s^2+2Rr+r^2)^2} \overset{?}{\geq}$$

$$17\sum_{cyc}a^2b^2-128r^2s^2 = 17((s^2+4Rr+r^2)^2-16Rrs^2)-128r^2s^2$$

$$\Leftrightarrow 15s^8-(316Rr+164r^2)s^6+r^2(1740R^2-2236Rr-634r^2)s^4-$$
$$r^3(2592R^3+3752R^2r+1748Rr^2+228r^3)s^2-$$
$$r^4(1088R^4+1632R^3r+884R^2r^2+204Rr^3+17r^4)\ \boxed{\overset{?}{\geq}\ (1)}\ 0 \text{ and}$$

$$\because 15(s^2-16Rr+5r^2)^4 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (1), \text{it suffices to prove :}$$

$$\text{LHS of } (1) \overset{?}{\geq} 15(s^2-16Rr+5r^2)^4$$

$$\Leftrightarrow (161R-116r)s^6-r(5325R^2-4159Rr+404r^2)s^4+$$
$$r^2(32(1899R^3-1830R^2r+549Rr^2-60r^3)+24R^3+22R^2r-5Rr^2-12r^3)s^2$$
$$-r^3\left(\begin{matrix}64(3844R^4-4794R^3r+2254R^2r^2-468Rr^3+36r^4)+\\ 16R^4+24R^3r-35R^2r^2+3Rr^3+44r^4\end{matrix}\right)\ \boxed{\overset{?}{\geq}\ (2)}\ 0 \text{ and } \because$$

$$P = (161R-116r)(s^2-16Rr+5r^2)^3+$$
$$r(2403R^2-3824Rr+1336r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{to prove } (2),$$

$$\text{suffices to prove : LHS of } (2) \overset{?}{\geq} P \Leftrightarrow (1755R^3-4821R^2r+3850Rr^2-824r^3)s^2$$
$$\boxed{\overset{?}{\geq}\ (3)}\ r\left(\begin{matrix}32(788R^4-22535R^3r+2029R^2r^2-703Rr^3+83r^4)+\\ 2R^4+15R^3r+11R^2r^2-2Rr^3\end{matrix}\right); \text{Finally, LHS of } (3)$$

$$\overset{\text{Rouche+Euler}}{\geq} (1755R^3-4821R^2r+3850Rr^2-824r^3)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right)$$

$$\overset{?}{\geq} \text{RHS}_{(3)} \Leftrightarrow 2862t^5-14913t^4+26241t^3-17210t^2+2876t+184 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\Big((t-2)(2862t^3-3465t^2+933t+382)+672\Big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \left(\sum_{\text{cyc}} \frac{a^2}{b+c}\right)^2 \geq \frac{17\sum_{\text{cyc}} a^2b^2 - 128r^2s^2}{4\sum_{\text{cyc}} a^2} \text{ \& implementing it on } \Delta\, XYZ \text{ with sides}$$

$$\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3} \text{ whose medians are } \frac{a}{2}, \frac{b}{2}, \frac{c}{2} \text{ \& area } \frac{F}{3}, \text{we get : } \frac{4}{9}\left(\sum_{\text{cyc}} \frac{m_a^2}{m_b+m_c}\right)^2$$

$$\geq \frac{17\sum_{\text{cyc}} a^2b^2 - 128F^2}{12\sum_{\text{cyc}} a^2} \left(\because \overbrace{\sum_{\text{cyc}} m_a^2m_b^2 = \frac{9\sum_{\text{cyc}} a^2b^2}{16} \text{ and } \sum_{\text{cyc}} m_a^2 = \frac{3\sum_{\text{cyc}} a^2}{4}}^{\sigma}\right)$$

$$= \frac{17\sum_{\text{cyc}} a^2b^2 - 8\left(2\sum_{\text{cyc}} a^2b^2 - \sum_{\text{cyc}} a^4\right)}{12\sum_{\text{cyc}} a^2} \Rightarrow \left(\sum_{\text{cyc}} \frac{m_a^2}{m_b+m_c}\right)^2 \geq$$

$$\frac{3\sum_{\text{cyc}}(8a^4+b^2c^2)}{16\sum_{\text{cyc}} a^2} \text{ \& also, } \frac{2\sum_{\text{cyc}} a^2b^2 - 23F^2}{\sum_{\text{cyc}} a^2} \overset{?}{\geq} \frac{1}{4}\left(\sum_{\text{cyc}} \frac{a^2}{b+c}\right)^2 \Leftrightarrow (16R-r)s^6 -$$

$$r(120R^2+196Rr+64r^2)s^4 + r^2(192R^3+212R^2r+68Rr^2+3r^3)s^2 +$$

$$r^3(128R^4+192R^3r+104R^2r^2+24Rr^3+2r^4) \boxed{\overset{?}{\geq}④} 0 \text{ and } \because Q =$$

$$(16R-r)(s^2-16Rr+5r^2)^3 + r(648R^2-484Rr-49r^2)(s^2-16Rr+5r^2)^2$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove ④, it suffices to prove : LHS of ④} \overset{?}{\geq} Q$$

$$\Leftrightarrow \begin{pmatrix} 2160R^3 - 3327R^2r + 415Rr^2 + \\ 142r^3 \end{pmatrix} s^2 \boxed{\overset{?}{\geq}⑤} r \begin{pmatrix} 25056R^4 - 40560R^3r + \\ 14488R^2r^2 - 271Rr^3 - 338r^4 \end{pmatrix}$$

$$\text{Finally, LHS}_{⑤} \overset{\text{Gerretsen}}{\geq} (2160R^3 - 3327R^2r + 415Rr^2 + 142r^3)(16Rr - 5r^2)$$

$$\overset{?}{\geq} \text{RHS of ⑤} \Leftrightarrow 3168t^4 - 7824t^3 + 2929t^2 + 156t - 124 \overset{?}{\geq} 0 \Leftrightarrow$$

$$(t-2)(3168t^3 - 1488t^2 - 47t + 62) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \frac{1}{4}\left(\sum_{\text{cyc}} \frac{a^2}{b+c}\right)^2 \leq$$

$$\frac{2\sum_{\text{cyc}} a^2b^2 - 23F^2}{\sum_{\text{cyc}} a^2} \text{ \& invoking on } \Delta\, XYZ \text{ using } ''\sigma'' \text{ \& } 2\sum_{\text{cyc}} a^2b^2 - \sum_{\text{cyc}} a^4 = 16F^2,$$

$$\left(\sum_{\text{cyc}} \frac{m_a^2}{m_b+m_c}\right)^2 \leq \frac{3\sum_{\text{cyc}}(23a^4 - 14b^2c^2)}{16\sum_{\text{cyc}} a^2}, '' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$