ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\left(\sum_{cyc} h_a\right)\left(\sum_{cyc} w_a\right)\left(\sum_{cyc} r_a\right) \le \prod_{cyc}(r_a + r_b + h_c)$$

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$$\prod_{cyc}\left(4R\cos^2\frac{A}{2} + h_a\right) = 64R^3.\frac{s^2}{16R^2} + \frac{2r^2s^2}{R} +$$

$$16R^2.\sum_{cyc}\left(\frac{s(s-b)}{ca}.\frac{s(s-c)}{ab}.\frac{bc}{2R}.\frac{b^2c^2}{b^2c^2}\right) + 4R.\sum_{cyc}\left(\frac{s(s-a)}{bc}.\frac{ca}{2R}.\frac{ab}{2R}\right)$$

$$= 4Rs^2 + \frac{2r^2s^2}{R} +$$

$$\frac{1}{2Rr^2}\left(\begin{matrix}-s^2((s^2+4Rr+r^2)^2 - 16Rrs^2) + (s^2+4Rr+r^2)^3 - \\ 24Rrs^2(s^2+2Rr+r^2) + 4Rrs^2(s^2+4Rr+r^2)\end{matrix}\right) +$$

$$\frac{s}{R}\left(2s(s^2-4Rr-r^2) - 2s(s^2-6Rr-3r^2)\right) \overset{?}{\ge}$$

$$\frac{(s^2+4Rr+r^2)(4R+r).\sqrt{s^2+21Rr+12r^2}}{2R} \overset{\text{squaring}}{\Leftrightarrow} s^8 + 27r^2s^6 +$$

$$(64R^4 - 272R^3r - 120R^2r^2 - 5Rr^3 + 184r^4)s^4 +$$

$$r(1024R^5 - 1664R^4r - 1440R^3r^2 - 296R^2r^3 - 2Rr^4 + 3r^5)s^2 -$$

$$r^2\left(\begin{matrix}4096R^6 + 768R^5r - 4608R^4r^2 - 3808R^3r^3 - 1248R^2r^4 - \\ 189Rr^5 - 11r^6\end{matrix}\right) \boxed{\overset{?}{\underset{①}{\ge}}} 0$$

Now, Rouche $\Rightarrow s^2 - (m-n) \ge 0$ *and* $s^2 - (m+n) \le 0$, where $m =$ $2R^2 + 10Rr - r^2$ & $n = 2(R-2r)\sqrt{R^2-2Rr}$ $\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right) \le 0$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0 \Rightarrow s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3 \le 0$$

$\therefore$ in order to prove ①, it suffices to prove :

$$\text{LHS of ①} \overset{?}{\ge} (s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3)^2 \Leftrightarrow$$

$$(8R^2+40Rr+23r^2)s^6 + (48R^4 - 560R^3r - 600R^2r^2 + 51Rr^3 + 178r^4)s^4 +$$

$$r(1536R^5 + 1280R^4r + 320R^3r^2 - 10Rr^4 - r^5)s^2 -$$

$$r^3(5376R^5 + 8448R^4r + 5088R^3r^2 + 1488R^2r^3 + 213Rr^4 + 12r^5) \boxed{\overset{?}{\underset{②}{\ge}}} 0 \text{ and } \because$$

$$(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove ②,}$$

$$\text{it suffices to prove : LHS of ②} \overset{?}{\ge} (8R^2+40Rr+23r^2)(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (48R^4 - 176R^3r + 1200R^2r^2 + 555Rr^3 - 167r^4)s^4 +$$

$$r(1536R^5 - 4864R^4r - 26560R^3r^2 + 936R^2r^3 + 8030Rr^4 - 1726r^5)s^2 +$$

$$r^3\begin{pmatrix}64(428R^5+1948R^4r-858R^3r^2-669R^2r^3+350Rr^4-45r^5)\\+32R^3r^2+8R^2r^3-13Rr^4-7r^5\end{pmatrix}\boxed{\overset{?}{\underset{③}{\geq}}}\,0 \text{ and } \because$$

$$P=(48R^4-176R^3r+1200R^2r^2+555Rr^3-167r^4)(s^2-16Rr+5r^2)^2$$

$$\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove ③, it suffices to prove : LHS of ③} \overset{?}{\geq} P$$

$$\Leftrightarrow (384R^5-1372R^4r+1700R^3r^2+837R^2r^3-358Rr^4-7r^5)s^2\,\boxed{\overset{?}{\underset{④}{\geq}}}$$

$$r(1536R^6-10016R^5r+26486R^4r^2+70R^3r^3-7343R^2r^4+2276Rr^5-161r^6)$$

$$\text{and finally, LHS of ④} \overset{\text{Gerretsen}}{\geq}$$

$$\begin{pmatrix}384R^5-1372R^4r+1700R^3r^2+837R^2r^3-\\358Rr^4-7r^5\end{pmatrix}(16Rr-5r^2)\overset{?}{\geq}\text{RHS}_{④}$$

$$\Leftrightarrow 2304t^6-6928t^5+3787t^4+2411t^3-1285t^2-299t+98\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(2304t^5-2320t^4-853t^3+705t^2+125t-49)\overset{?}{\geq}0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow ④ \Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$\therefore \prod_{\text{cyc}}(r_a+r_b+h_c)\geq\frac{(s^2+4Rr+r^2)(4R+r).\sqrt{s^2+21Rr+12r^2}}{2R}$$

$$\overset{\text{Dang Ngoc Minh}}{\geq}\left(\sum_{\text{cyc}}h_a\right)\left(\sum_{\text{cyc}}r_a\right)\left(\sum_{\text{cyc}}w_a\right)$$

(**Reference : Inequality in Triangle by Dang Ngoc Minh – 90; published at www.ssmrmh.ro**) *and so,*

$$\left(\sum_{\text{cyc}}h_a\right)\left(\sum_{\text{cyc}}w_a\right)\left(\sum_{\text{cyc}}r_a\right)\leq\prod_{\text{cyc}}(r_a+r_b+h_c)\ \forall\, ABC,$$

$$''=''\text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$