ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$m_a \le \sqrt{\frac{R+2r}{16r}}.(r_b+r_c)$$

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Let $x = s-a, y = s-b, z = s-c$ and then : $a = y+z, b = z+x,$
$c = x+y$ and $s = x+y+z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$
and then, we have the following set $''S''$ of relations : $y^2+z^2 = x^2(m^2-2n),$
$y^3+z^3 = x^3(m^3-3nn), y^4+z^4 = x^4((m^2-2n)^2-2n^2),$

$$y^5+z^5 = x^5\left(m((m^2-2n)^2+n^2-nm^2)\right), \text{ and now, } m_a \overset{?}{\le} \sqrt{\frac{R+2r}{16r}}.(r_b+r_c)$$

$$\Leftrightarrow (b-c)^2+4s(s-a) \overset{?}{\le} \left(\frac{R}{4r}+\frac{1}{2}\right).\frac{16R^2s^2(s-a)^2a^2}{16R^2.s(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow (y-z)^2+4x(x+y+z) \overset{?}{\le} \frac{(y+z)(z+x)(x+y)+8xyz}{16xyz}.\frac{x(x+y+z)(y+z)^2}{yz}$$

$$\Leftrightarrow x^3(y+z)^3+2x^2(y^4+z^4)+16x^2.yz(y^2+z^2)-36x^2.y^2z^2+x\left(y^5+z^5\right)+$$
$$14x.yz(y^3+z^3)-27x.y^2z^2(y+z)+yz(y^4+z^4)-12y^2z^2(y^2+z^2)+38y^3z^3$$
$$\boxed{\overset{?}{\underset{(1)}{\ge}}} 0$$ and via set of relations $''S''$, in order to prove (1), it suffices to prove,

following simplification : $m^5+m^4n+2m^4+9m^3n+m^3+8m^2n-64mn^2-$
$64n^2+16n^2(4n-m^2) \overset{?}{\ge} 0$ and $\because 16n^2(4n-m^2) \ge 4nm^2(4n-m^2)$
$\therefore$ it suffices to prove :

$$m^5+m^4n+2m^4+9m^3n+m^3+8m^2n-64mn^2-64n^2+4nm^2(4n-m^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow \overbrace{(16m^2-64m-64)}^{\Omega_1}n^2-\overbrace{(3m^4-9m^3-8m^2)}^{\Omega_2}n+\overbrace{m^3(m+1)^2}^{\Omega_3} \boxed{\overset{?}{\underset{(2)}{\ge}}} 0$$

$\boxed{\text{Case 1}}$ $m^2-4m-4>0$ and now, LHS of (2) is a quadratic polynomial in
$''n''$ with discriminant, $\delta = \Omega_2^2-4\Omega_1.\Omega_3 =$
$m^3\left(9m^5-118m^4+161m^3+848m^2+832m+256\right)$ and when $\delta \le 0$,
then (2) is trivially true and when : $\delta > 0$, then it suffices to prove :

$$2\Omega_1.n \overset{?}{\le} \Omega_2-\sqrt{\delta} \text{ and } \because n \overset{AM-GM}{\le} \frac{m^2}{4} \therefore \text{ it suffices to prove : } m^2.\Omega_1 \overset{?}{\le} 2\Omega_2-2\sqrt{\delta}$$

$$\Leftrightarrow 3m^4-9m^3-16m^2+32m+32 \overset{?}{\ge} \sqrt{\delta} \text{ and } \because m^2-4m>4>0 \therefore \boxed{m>4}$$
$$\Rightarrow 3m^4-9m^3-16m^2+32m+32 =$$
$$(m-4)(3m^3+2m^2+m(m-4)+16)+96>0 \therefore \text{ it suffices to prove :}$$
$$(3m^4-9m^3-16m^2+32m+32)^2 \overset{?}{\ge} m^3\begin{pmatrix}9m^5-118m^4+161m^3+\\848m^2+832m+256\end{pmatrix}$$

$$\Leftrightarrow \boxed{64(m+1)(m^2-4m-4)\left(4m^4+m^3+11m^2+(m-4)(m+4)\right) \overset{?}{\geq} 0}$$

$\to$ true (strict inequality) $\Rightarrow$ 2 $\Rightarrow$ 1 is true

$\boxed{Case\ 2}$ $m^2-4m-4<0$ *and then*, 2 *can be re – written as* :

$$\overbrace{(64+64m-16m^2)}^{-\Omega_1}n^2+\overbrace{(3m^4-9m^3-8m^2)}^{\Omega_2}n-\overbrace{m^3(m+1)^2}^{\Omega_3}\ \boxed{\underset{(3)}{\overset{?}{\leq}}}\ 0$$

Discriminant of LHS of 3 is again, $\delta=\Omega_2^2-4(-\Omega_1).(-\Omega_3)$

$$=m^3(9m^5-118m^4+161m^3+848m^2+832m+256)$$

$$=\frac{(m^3(9m-44)-30m^3)(9m-44)}{9}-\left(\frac{1807m^2}{81}+\frac{10820m}{729}\right)(9m-44)+$$

$$178m+\frac{686m}{729}+256>0\ \left(\because m^2-4m-4<0\Rightarrow 0<m<2+2\sqrt{2}<\frac{44}{9}\right)$$

$\therefore$ it suffices to prove : $2(-\Omega_1).n\ \boxed{\underset{(*)}{\overset{?}{\leq}}}\ -\Omega_2+\sqrt{\delta}$ AND $2(-\Omega_1).n\ \boxed{\underset{(**)}{\overset{?}{\geq}}}\ -\Omega_2-\sqrt{\delta}$

Now, $2(-\Omega_1).n\leq\frac{(-\Omega_1).m^2}{2}\overset{?}{\leq}-\Omega_2+\sqrt{\delta}\Leftrightarrow m^2(24+23m-5m^2)\overset{?}{\leq}\sqrt{\delta}$ *and*

$\because 24+23m-5m^2=(5-m)(5m+2)+14>0\ (\because m<2+2\sqrt{2}<5)$

$\therefore$ it suffices to prove :

$$m^3(9m^5-118m^4+161m^3+848m^2+832m+256)\overset{?}{\geq}m^4(24+23m-5m^2)^2$$

$$\Leftrightarrow \boxed{16m^3(m+1)(m-2)^2(4+4m-m^2)\overset{?}{\geq}0}\to \text{true}\Rightarrow (*)\text{ is true}$$

and $2(-\Omega_1).n+\sqrt{\delta}>\sqrt{\delta}=\sqrt{\Omega_2^2+4(-\Omega_1).\Omega_3}\overset{-\Omega_1>0}{>}\sqrt{\Omega_2^2}=|\Omega_2|\geq-\Omega_2$

$\Rightarrow (**)$ is true *and so*, 3 $\Rightarrow$ 2 $\Rightarrow$ 1 is true

$$\therefore m_a\leq\sqrt{\frac{R+2r}{16r}}.(r_b+r_c)\ \forall\ \Delta\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$