ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{2r_a}{h_b+h_c}+\frac{m_b+m_c}{2r_a}\le\frac{R}{r}$$

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$$We\ will\ show:\ a^2-a(b+c)+4bc>0\ in\ any\ \Delta ABC\ (1)$$

$$Proof: we\ know\ that\ a>|b-c|then\ a^2>(b-c)^2$$

$$a^2-a(b+c)+4bc>(b-c)^2-a(b+c)+4bc=b^2+2bc+c^2-a(b+c)$$
$$=(b+c)^2-a(b+c)=(b+c)(b+c-a)>0$$

$$as\ in\ \Delta ABC\ ,b+c>a(proof\ complete)$$

$$\frac{m_b+m_c}{2r_a}\overset{Panaitopol}{\le}\frac{R}{4r}\left(\frac{h_b+h_c}{r_a}\right)$$

$$we\ will\ show: \frac{2r_a}{h_b+h_c}+\frac{m_b+m_c}{2r_a}\le\frac{R}{r}\ \ or,\frac{2r_a}{h_b+h_c}+\frac{R}{4r}\left(\frac{h_b+h_c}{r_a}\right)\le\frac{R}{r}\ \ (2)$$

$$let\ x=\frac{2r_a}{h_b+h_c}=\frac{\frac{2F}{s-a}}{\frac{2F}{b}+\frac{2F}{c}}=\frac{bc}{(b+c)(s-a)}\ then\ \frac{h_b+h_c}{r_a}=\frac{2}{x}$$

$$from\ (2)we\ get\ ,x+\frac{R}{4r}\left(\frac{2}{x}\right)\le\frac{R}{r}\ or,\frac{2r}{R}x^2-2x+1\le 0\ (3)$$

$$\frac{r}{R}=\frac{4F^2}{abcs}=\frac{4(s-a)(s-b)(s-c)}{abc}$$

$$putting\ the\ value\ of\ x\ and\ \frac{r}{R}\ in(3)we\ get$$

$$2\left(\frac{4(s-a)(s-b)(s-c)}{abc}\right)\left(\frac{bc}{(b+c)(s-a)}\right)^2-2\frac{bc}{(b+c)(s-a)}+1\le 0$$

$$or, 8bc(s-b)(s-c)-2abc(b+c)+a(b+c)^2(s-a)\le 0$$

$$or,2bc(a^2-(b-c)^2)-2abc(b+c)+a(b+c)^2(s-a)\leq 0$$

$$or,4bc(a^2-(b-c)^2)-4abc(b+c)+a(b+c)^2(2s-2a)\leq 0(Multiply\ by\ 2)$$

$$or,4bc(a^2-(b-c)^2)-4abc(b+c)+a(b+c)^2(b+c-a)\leq 0$$

$$or,a^2(4bc-(b+c)^2)+\left((b+c)^3-4bc(b+c)\right)a-4bc(b-c)^2\leq 0$$

$$or,-a^2(b-c)^2+a(b+c)(b-c)^2-4bc(b-c)^2\leq 0$$

$$or,-(b-c)^2(a^2-a(b+c)+4bc)\leq 0\ true\ by\ (1)$$

Equality holds for b=c