ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sqrt{\frac{r_a}{h_a}}+\sqrt{\frac{r_b+r_c}{2r_a}}\le\sqrt{\frac{2R}{r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$For\ \ x=s-a,y=s-b,z=s-c\ then\ a=y+z,b=z+x,c=x+y$$

$$r_a=\frac{F}{s-a}=\frac{F}{x},similarly,r_b=\frac{F}{y},r_c=\frac{F}{z}\ \&\ h_a=\frac{2F}{a}=\frac{2F}{y+z}$$

$$\frac{r_a}{h_a}=\frac{y+z}{2x},\frac{r_b+r_c}{2r_a}=\frac{x(y+z)}{2yz},$$

$$using\ the\ above\ result\ we\ get,L.H.S=\sqrt{\frac{y+z}{2x}}+\sqrt{\frac{x(y+z)}{2yz}}=\sqrt{\frac{y+z}{2x}}\left(1+\frac{x}{\sqrt{yz}}\right)$$

$$\frac{R}{r}=\frac{\frac{abc}{4F}}{\frac{F}{s}}=\frac{abcs}{4F^2}=\frac{abcs}{4s(s-a)(s-b)(s-c)}=\frac{(x+y)(y+z)(z+x)}{4xyz}$$

$$or,\frac{2R}{r}=\frac{(x+y)(y+z)(z+x)}{2xyz}$$

$$so\ it\ is\ sufficient\ to\ show:$$

$$\sqrt{\frac{y+z}{2x}}\left(1+\frac{x}{\sqrt{yz}}\right)\le\sqrt{\frac{(x+y)(y+z)(z+x)}{2xyz}}\ or,\left(1+\frac{x}{\sqrt{yz}}\right)\le\sqrt{\frac{(x+y)(z+x)}{yz}}$$

$$or,\left(1+\frac{x}{\sqrt{yz}}\right)^2\le\frac{(x+y)(x+z)}{yz}\ or,1+\frac{2x}{\sqrt{yz}}+\frac{x^2}{yz}\le 1+\frac{x}{y}+\frac{x}{z}+\frac{x^2}{yz}\ or,\frac{1}{y}+\frac{1}{z}\ge\frac{2}{\sqrt{yz}}$$

$(True\ by\ AM-GM).\ Equality\ holds\ for\ y=z\ or,b=c$