ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC holds :

$$w_a w_b + w_b w_c + w_c w_a \geq \sqrt{3\left(g_a^2 g_b^2 + g_b^2 g_c^2 + g_c^2 g_a^2\right)}$$

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$$\left(\sum_{cyc} w_b w_c\right)^2 = \sum_{cyc} w_b^2 w_c^2 + 2 w_a w_b w_c \left(\sum_{cyc} w_a\right) \geq$$

$$\frac{256Rr^4s^4}{(s^2+2Rr+r^2)^2} \cdot \frac{(2R+r)s^2 + r^2(4R+r)}{8Rr^2s^2} + \frac{32Rr^2s^2}{s^2+2Rr+r^2}\left(\sum_{cyc} h_a\right)$$

$$\left(\because \sum_{cyc} \frac{1}{w_a^2} = \frac{(2R+r)s^2 + r^2(4R+r)}{8Rr^2s^2}; \text{Reference : Solution to Inequality in Triangle by Dang Ngoc Minh} - 258; \text{published at www.ssmrmh.ro}\right)$$

$$= \frac{32Rr^2s^2\left((2R+r)s^2 + r^2(4R+r)\right) + 16r^2s^2(s^2+2Rr+r^2)(s^2+4Rr+r^2)}{(s^2+2Rr+r^2)^2} \overset{?}{\geq} 3\sum_{cyc} g_b^2 g_c^2$$

$$= \frac{3r^2\left((6R+6r)s^2 + 16R^3 - 24R^2r - 15Rr^2 - 2r^3\right)}{R}$$

$$\left(\text{Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba} - 69; \text{published at www.ssmrmh.ro}\right)$$

$$\Leftrightarrow -(2R+18r)s^6 + (16R^3 + 128R^2r - 31Rr^2 - 30r^3)s^4 - r(192R^4 - 376R^3r - 308R^2r^2 - 40Rr^3 + 6r^4)s^2 - r^2(192R^5 - 96R^4r - 420R^3r^2 - 276R^2r^3 - 69Rr^4 - 6r^5) \boxed{\overset{?}{\underset{①}{\geq}}} 0$$

Now, Rouche $\Rightarrow s^2 - (m-n) \geq 0$ and $s^2 - (m+n) \leq 0$, where $m = 2R^2 + 10Rr - r^2$ & $n = 2(R-2r)\sqrt{R^2-2Rr}$ $\therefore \left(s^2 - (m+n)\right)\left(s^2 - (m-n)\right) \leq 0$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \overset{(*)}{\leq} 0$$

$$\Rightarrow P = -(2R+18r)s^2(s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3) \geq 0$$

$\therefore$ in order to prove ①, it suffices to prove : LHS of ① $\overset{?}{\geq}$ P

$$\Leftrightarrow (8R^3 + 16R^2r - 387Rr^2 + 6r^3)s^4 - r(64R^4 - 1624R^3r - 1196R^2r^2 - 258Rr^3 - 12r^4)s^2 - r^2(192R^5 - 96R^4r - 420R^3r^2 - 276R^2r^3 - 69Rr^4 - 6r^5) \boxed{\overset{?}{\underset{②}{\geq}}} 0$$

$\boxed{\text{Case 1}}$ $8R^3 + 16R^2r - 387Rr^2 + 6r^3 \geq 0$ and then, since Q =

$$(8R^3+16R^2r-387Rr^2+6r^3)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$\therefore$ in order to prove ②, it suffices to prove : LHS of ② $\overset{?}{\geq}$ Q

$$\Leftrightarrow (48R^4+514R^3r-2837R^2r^2+1080Rr^3-12r^4)s^2 \boxed{\overset{?}{\geq} ③}$$

$$r\left(560R^5+680R^4r-25463R^3r^2+15895R^2r^3-2676Rr^4+36r^5\right)$$

Now, with $t=\frac{R}{r}$, $8t^3+16t^2-387t+6\geq 0 \Rightarrow (t-6)(8t^2+64t-3)\geq 12>0$

$$\Rightarrow t>6 \Rightarrow 48t^4+514t^3-2837t^2+1080t-12$$

$=(t-6)(48t^3+802t^2+1975t+12930)+77568>0 \therefore$ LHS of ③ $\overset{\text{Gerretsen}}{\geq}$

$(48R^4+514R^3r-2837R^2r^2+1080Rr^3-12r^4)(16Rr-5r^2) \overset{?}{\geq}$ RHS of ③

$$\Leftrightarrow 208t^5+7304t^4-22499t^3+15570t^2-2916t+24 \overset{?}{\geq} 0 \Leftrightarrow$$

$(t-2)(208t^4+7720t^3-7059t^2+1452t-12) \overset{?}{\geq} 0$; true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ ③ is true

$\boxed{\text{Case 2}}$ $8R^3+16R^2r-387Rr^2+6r^3<0$ *and* then, since T =

$$(8R^3+16R^2r-387Rr^2+6r^3)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3) \overset{\text{via }(*)}{\geq} 0$$

$\therefore$ in order to prove ②, it suffices to prove : LHS of ② $\overset{?}{\geq}$ T

$$\Leftrightarrow (8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4)s^2 \boxed{\overset{?}{\geq} ④}$$

$$r\left(128R^5+400R^4r-6000R^3r^2-4603R^2r^3-1154Rr^4-96r^5\right)$$

$\boxed{\text{Case }(2a)}$ $8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4\geq 0$; then : $\text{LHS}_{④} \overset{\text{Gerretsen}}{\geq}$

$(8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4)(16Rr-5r^2) \overset{?}{\geq} \text{RHS}_{④}$

$$\Leftrightarrow 25t^4+915t^3-2760t^2+1744t-168 \overset{?}{\geq} 0 \Leftrightarrow$$

$(t-2)(25t^3+965t^2-830t+84) \overset{?}{\geq} 0 \to$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow$ ④ is true

$\boxed{\text{Case }(2b)}$ $8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4<0$; then : $\text{LHS}_{④} \overset{\text{Gerretsen}}{\geq}$

$(8R^4+40R^3r+95R^2r^2-1638Rr^3+288r^4)(4R^2+4Rr+3r^2) \overset{?}{\geq} \text{RHS}_{④}$

$$\Leftrightarrow 8t^6+16t^5+41t^4-13t^3-128t^2-652t+240 \overset{?}{\geq} 0 \Leftrightarrow$$

$(t-2)\left(8t^5+32t^4+105t^3+197t^2+266t-120\right) \overset{?}{\geq} 0 \to$ true $\because t \overset{\text{Euler}}{\geq} 2$

$\Rightarrow$ ④ is true $\therefore$ combining *cases* $(2a)$ *and* $(2b)$, ④ $\Rightarrow$ ② $\Rightarrow$ ① is true

$\forall\,\Delta$ ABC *and* so, $w_a w_b+w_b w_c+w_c w_a \geq \sqrt{3\left(g_a^2g_b^2+g_b^2g_c^2+g_c^2g_a^2\right)}\ \forall$ ABC,

" = " iff Δ ABC is equi*lateral* (QED)