ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{bc}{a^2}-\frac{r}{R}-\frac{1}{4}\geq\sum_{cyc}\frac{w_a^2}{a^2}\geq\sum_{cyc}\frac{bc}{a^2}+\frac{r}{9R}-\frac{29}{36}$$

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$$\sum_{cyc}\frac{w_a^2}{a^2}=\sum_{cyc}\frac{4bc.s(s-a)}{a^2(b+c)^2}=\sum_{cyc}\frac{bc((b+c)^2-a^2)}{a^2(b+c)^2}=\sum_{cyc}\frac{bc}{a^2}-\sum_{cyc}\frac{bc}{(b+c)^2}$$

$$=\sum_{cyc}\frac{bc}{a^2}-\frac{1}{4s^2(s^2+2Rr+r^2)^2}.\sum_{cyc}\left(bc\left(a^4+2a^2\sum_{cyc}ab+\left(\sum_{cyc}ab\right)^2\right)\right)$$

$$=\sum_{cyc}\frac{bc}{a^2}-\frac{1}{4s^2(s^2+2Rr+r^2)^2}.\left(\begin{matrix}4Rrs.2s(s^2-6Rr-3r^2)+\\2(s^2+4Rr+r^2).4Rrs.2s+(s^2+4Rr+r^2)^3\end{matrix}\right)$$

$$\overset{?}{\leq}\sum_{cyc}\frac{bc}{a^2}-\frac{r}{R}-\frac{1}{4}\Leftrightarrow-4s^6+(32R^2-15Rr-8r^2)s^4+$$

$$r(60R^3-4R^2r-14Rr^2-4r^3)s^2+Rr^2(4R+r)^3\ \boxed{\overset{?}{\geq}}_{(1)}\ 0$$

$$\text{Now, Rouche}\Rightarrow s^2-(m-n)\geq 0\ \textit{and}\ s^2-(m+n)\leq 0,\text{where } m=$$

$$2R^2+10Rr-r^2\ \&\ n=2(R-2r)\sqrt{R^2-2Rr}\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(*)}{\leq}0$$

$$\Rightarrow P=-4s^2(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0$$

$$\therefore\text{ in order to prove (1), it suffices to prove : LHS of (1)}\overset{?}{\geq}P$$

$$\Leftrightarrow(16R-95r)s^4+r(316R^2+188Rr+34r^2)s^2+r^2(4R+r)^3\ \boxed{\overset{?}{\geq}}_{(2)}\ 0$$

and it's *trivially* true if : $16R-95r\geq 0$ *and* when : $16R-95r<0$, then via $(*)$,

$$Q=(16R-95r)(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)\geq 0\therefore\text{ to prove (2),}$$

$$\text{it suffices to prove : LHS of (2)}\overset{?}{\geq}Q\Leftrightarrow$$

$$(4R^3+16R^2r-109Rr^2+14r^3)s^2 \boxed{\overset{?}{\underset{(3)}{\geq}}} r\begin{pmatrix}64R^4-336R^3r-276R^2r^2- \\ 71Rr^3-6r^4\end{pmatrix}$$

$\boxed{\text{Case 1}}$ $4R^3+16R^2r-109Rr^2+14r^3 \geq 0$ *and then* : LHS of (3) $\overset{\text{Gerretsen}}{\geq}$

$$(4R^3+16R^2r-109Rr^2+14r^3)(16Rr-5r^2) \overset{?}{\geq} \text{RHS of (3)}$$

$$\Leftrightarrow 143t^3-387t^2+210t-16 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(143t^2-101t+8) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (3) \text{ is true}$$

$\boxed{\text{Case 2}}$ $4R^3+16R^2r-109Rr^2+14r^3 < 0$ *and then* : LHS of (3) $\overset{\text{Gerretsen}}{\geq}$

$$(4R^3+16R^2r-109Rr^2+14r^3)(4R^2+4Rr+3r^2) \overset{?}{\geq} \text{RHS of (3)}$$

$$\Leftrightarrow 2t^5+2t^4-3t^3-7t^2-25t+6 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(2t^4+6t^3+9t^2+11t-3) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (3) \text{ is true} \therefore \text{combining both cases,}$$

$$(3) \Rightarrow (2) \Rightarrow (1) \text{ is true } \forall \Delta ABC \therefore \sum_{cyc}\frac{w_a^2}{a^2} \leq \sum_{cyc}\frac{bc}{a^2}-\frac{r}{R}-\frac{1}{4}$$

$$\text{Again, } \sum_{cyc}\frac{w_a^2}{a^2} = \sum_{cyc}\frac{bc}{a^2}-\sum_{cyc}\frac{bc}{(b+c)^2} \overset{\text{AM-GM}}{\geq} \sum_{cyc}\frac{bc}{a^2}-\sum_{cyc}\frac{bc}{4bc} = \sum_{cyc}\frac{bc}{a^2}-\frac{3}{4}$$

$$\overset{?}{\geq} \sum_{cyc}\frac{bc}{a^2}+\frac{r}{9R}-\frac{29}{36} \Leftrightarrow \frac{1}{18} \overset{?}{\geq} \frac{r}{9R} \Leftrightarrow R \overset{?}{\geq} 2r \to \text{true via Euler}$$

$$\therefore \sum_{cyc}\frac{w_a^2}{a^2} \geq \sum_{cyc}\frac{bc}{a^2}+\frac{r}{9R}-\frac{29}{36} \text{ and so,}$$

$$\sum_{cyc}\frac{bc}{a^2}-\frac{r}{R}-\frac{1}{4} \geq \sum_{cyc}\frac{w_a^2}{a^2} \geq \sum_{cyc}\frac{bc}{a^2}+\frac{r}{9R}-\frac{29}{36} \; \forall \text{ ABC,}$$

$$'' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$