ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC holds :

$$\sqrt{\sum_{cyc}\frac{bc}{a^2}+\frac{31}{4}-\frac{8r}{R}} \geq \sum_{cyc}\frac{w_a}{a} \geq \sqrt{\sum_{cyc}\frac{bc}{a^2}+\frac{103}{36}+\frac{16r}{9R}}$$

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$$\sum_{cyc}\left(a^2\cos^2\frac{B-C}{2}\right)=\frac{1}{4Rrs}\cdot\sum_{cyc}\left(a(b+c)^2(-s^2+sa+bc)\right)$$

$$=\frac{1}{4Rrs}\cdot\left(\begin{matrix}-s^2(2s(s^2+4Rr+r^2)+12Rrs)+2s((s^2+4Rr+r^2)^2-8Rrs^2)+\\ 8Rrs(2(s^2-4Rr-r^2)+s^2+4Rr+r^2)\end{matrix}\right)$$

$$=\frac{(2R+r)s^2+r^2(4R+r)}{2R}\Rightarrow 2\sum_{cyc}\frac{w_bw_c}{bc}=2.\sum_{cyc}\frac{\frac{2ca}{c+a}\cos\frac{B}{2}.\frac{2ab}{a+b}\cos\frac{C}{2}}{bc}$$

$$=\frac{4\sum_{cyc}\frac{a^2\cos^2\frac{B-C}{2}}{\cos\frac{B-C}{2}}}{s^2+2Rr+r^2}\overset{0<\cos\frac{B-C}{2}\leq 1}{\geq}\frac{4\sum_{cyc}\left(a^2\cos^2\frac{B-C}{2}\right)}{s^2+2Rr+r^2}=\frac{4.\frac{(2R+r)s^2+r^2(4R+r)}{2R}}{s^2+2Rr+r^2}$$

$$\Rightarrow 2\sum_{cyc}\frac{w_bw_c}{bc}\geq\frac{2\left((2R+r)s^2+r^2(4R+r)\right)}{R(s^2+2Rr+r^2)}\therefore\left(\sum_{cyc}\frac{w_a}{a}\right)^2\geq\sum_{cyc}\frac{bc}{a^2}-$$

$$\frac{\sum_{cyc}\left(bc\left(a^4+2a^2\sum_{cyc}ab+\left(\sum_{cyc}ab\right)^2\right)\right)}{4s^2(s^2+2Rr+r^2)^2}+\frac{2\left((2R+r)s^2+r^2(4R+r)\right)}{R(s^2+2Rr+r^2)}$$

$$=\sum_{cyc}\frac{bc}{a^2}-\frac{1}{4s^2(s^2+2Rr+r^2)^2}\cdot\left(\begin{matrix}4Rrs.2s(s^2-6Rr-3r^2)+\\ 2(s^2+4Rr+r^2).4Rrs.2s+(s^2+4Rr+r^2)^3\end{matrix}\right)+$$

$$\frac{2\left((2R+r)s^2+r^2(4R+r)\right)}{R(s^2+2Rr+r^2)}\overset{?}{\geq}\sum_{cyc}\frac{bc}{a^2}+\frac{103}{36}+\frac{16r}{9R}\Leftrightarrow(32R+8r)s^6-$$

$$r(448R^2-87Rr-16r^2)s^4-r^2(988R^3+236R^2r-46Rr^2-8r^3)s^2-$$

$$9Rr^3(4R+r)^3\ \boxed{\overset{?}{\underset{(1)}{\geq}}}\ 0\ \textit{and}\ \because P=(32R+8r)(s^2-16Rr+5r^2)^3+$$

$$r(1088R^2-9Rr-104r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore\text{ in order to prove (1),}$$

$$\text{it suffices to prove : LHS of (1)}\overset{?}{\geq}P\Leftrightarrow(2313R^3-547R^2r-438Rr^2+112r^3)s^2$$

$$\boxed{\overset{?}{\underset{(2)}{\geq}}}\ r(37008R^4-21460R^3r-1389R^2r^2+2706Rr^3-400r^4)$$

$$\text{LHS of (2)}\overset{\text{Gerretsen}}{\geq}(2313R^3-547R^2r-438Rr^2+112r^3)(16Rr-5r^2)\overset{?}{\geq}$$

$$\text{RHS of (2)}\Leftrightarrow 1143t^3-2884t^2+1276t-160\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1143t^2 - 598t + 80) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$w_a \geq h_a \therefore \left(\sum_{\text{cyc}} \frac{w_a}{a}\right)^2 \leq \sum_{\text{cyc}} \frac{bc}{a^2} - \sum_{\text{cyc}} \frac{bc}{(b+c)^2} + \frac{2.16Rr^2s^2}{(s^2+2Rr+r^2)(4Rrs)} \cdot \sum_{\text{cyc}} \frac{a^2}{2rs}$$

$$= \sum_{\text{cyc}} \frac{bc}{a^2} - \frac{8Rrs^2(s^2-6Rr-3r^2) + (s^2+4Rr+r^2).16Rrs^2 + (s^2+4Rr+r^2)^3}{4s^2(s^2+2Rr+r^2)^2} +$$

$$\frac{8(s^2-4Rr-r^2)}{s^2+2Rr+r^2} \overset{?}{\leq} \sum_{\text{cyc}} \frac{bc}{a^2} + \frac{31}{4} - \frac{8r}{R} \Leftrightarrow -32s^6 + (224R^2 - 63Rr - 64r^2)s^4 +$$

$$r(444R^3 + 204R^2r - 62Rr^2 - 32r^3)s^2 + Rr^2(4R+r)^3 \boxed{\overset{?}{\underset{③}{\geq}}} 0$$

$$\text{Now, } Q = -32s^2(s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3) \overset{\text{Double-Rouche}}{\geq} 0$$

$\therefore$ in order to prove ③, it suffices to prove : LHS of ③ $\overset{?}{\geq} Q \Leftrightarrow (96R - 703r)s^4 +$

$r(2492R^2 + 1740Rr + 322r^2)s^2 + r^2(4R+r)^3 \boxed{\overset{?}{\underset{④}{\geq}}} 0$ *and* it's trivi*ally* true if :

$96R - 703r \geq 0$ *and* when : $96R - 703r < 0$, then via $(*)$, $Q' =$
$(96R - 703r)(s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3) \geq 0 \therefore$ to prove ④,

it suffices to prove : LHS of ④ $\overset{?}{\geq} Q' \Leftrightarrow (12R^3 + 50R^2r - 391Rr^2 + 54r^3)s^2$

$$\boxed{\overset{?}{\underset{⑤}{\geq}}} r(192R^4 - 1264R^3r - 1020R^2r^2 - 261Rr^3 - 22r^4)$$

$\boxed{\textit{Case 1}}$ $12R^3 + 50R^2r - 391Rr^2 + 54r^3 \geq 0$ *and* then : LHS of ⑤ $\overset{\text{Gerretsen}}{\geq}$

$$(12R^3 + 50R^2r - 391Rr^2 + 54r^3)(16Rr - 5r^2) \overset{?}{\geq} \text{RHS of ⑤}$$

$$\Leftrightarrow 1002t^3 - 2743t^2 + 1540t - 124 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(1002t^2 - 739t + 62) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ⑤ \text{ is true}$$

$\boxed{\textit{Case 2}}$ $12R^3 + 50R^2r - 391Rr^2 + 54r^3 < 0$ *and* then : LHS of ⑤ $\overset{\text{Gerretsen}}{\geq}$

$$(12R^3 + 50R^2r - 391Rr^2 + 54r^3)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} \text{RHS of ⑤}$$

$$\Leftrightarrow 24t^5 + 28t^4 - 32t^3 - 89t^2 - 348t + 92 \overset{?}{\geq} 0 \Leftrightarrow$$

$$(t-2)(24t^4 + 76t^3 + 120t^2 + 151t - 46) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ⑤ \text{ is true}$$

$$\therefore \sqrt{\sum_{\text{cyc}} \frac{bc}{a^2} + \frac{31}{4} - \frac{8r}{R}} \geq \sum_{\text{cyc}} \frac{w_a}{a} \geq \sqrt{\sum_{\text{cyc}} \frac{bc}{a^2} + \frac{103}{36} + \frac{16r}{9R}}, \text{"=" iff } a = b = c \text{ (QED)}$$