In any Δ ABC the following relationship holds :

$$(n_a+n_b+n_c)(m_a+m_b+m_c) \ge 3(m_a w_a + m_b w_b + m_c w_c)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{4s(s-a)}{(b+c)^2} = \sum_{cyc}\frac{(b+c)^2-a^2}{(b+c)^2} = 3 - \sum_{cyc}\frac{(2s-(2s-a)^2)}{(2s-a)^2}$$

$$= 4s\sum_{cyc}\frac{1}{b+c} - 4s^2\left(\left(\sum_{cyc}\frac{1}{b+c}\right)^2 - \frac{2}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}(b+c)\right)$$

$$= \frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)} - \frac{(5s^2+4Rr+r^2)^2}{(s^2+2Rr+r^2)^2} + \frac{16s^2}{s^2+2Rr+r^2}$$

$$\Rightarrow \sum_{cyc}\frac{4s(s-a)}{(b+c)^2} \boxed{\overset{(\bullet)}{=}} \frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{(s^2+2Rr+r^2)^2}$$

$$\text{Now, } 3\sum_{cyc} m_a w_a = 6\sum_{cyc}\left(m_a\cdot\frac{\sqrt{bcs(s-a)}}{b+c}\right) \overset{CBS}{\le}$$

$$3.\sqrt{\left(\sum_{cyc}\left(2\sum_{cyc}a^2-3a^2\right)bc\right)}\cdot\sqrt{\sum_{cyc}\frac{s(s-a)}{(b+c)^2}} \overset{via\ (\bullet)}{=}$$

$$3.\sqrt{4(s^4-(4Rr+r^2)^2-6Rrs^2)\cdot\frac{s^4+(20Rr+18r^2)s^2+r^3(4R+r)}{4(s^2+2Rr+r^2)^2}} \overset{?}{\le}$$

$$\sqrt{(9s^2-80Rr-2r^2)(4s^2-28Rr+29r^2)}$$

$$\overset{squaring}{\Leftrightarrow} 27s^8-(554Rr-163r^2)s^6+r^2(1320R^2-1244Rr+484r^2)s^4+$$
$$r^3(9552R^3-1604R^2r-2790Rr^2+299r^3)s^2+$$
$$r^4(8960R^4+480R^3r-6616R^2r^2-2388Rr^3-49r^4) \boxed{\overset{?}{\ge}\ (1)}\ 0 \text{ and } \because$$

$$27(s^2-16Rr+5r^2)^4 \overset{Gerretsen}{\ge} 0 \therefore \text{ in order to prove } (1), \text{ it suffices to prove :}$$

$$\text{LHS of } (1) \overset{?}{\ge} 27(s^2-16Rr+5r^2)^4$$

$$\Leftrightarrow (1174R-377r)s^6-r(40152R^2-24676Rr+3566r^2)s^4+$$
$$r^2(16(28245R^3-26021R^2r+7925Rr^2-825r^3)+12R^2r+10Rr^2-r^3)s^2-$$
$$r^3\left(\begin{matrix}128(13754R^4-17284R^3r+8152R^2r^2-1669Rr^3+132r^4)+\\ 32R^3r-40R^2r^2+20Rr^3+28r^4\end{matrix}\right) \boxed{\overset{?}{\ge}\ (2)}\ 0$$

$$\text{and } \because (1174R-377r)(s^2-16Rr+5r^2)^3 \overset{Gerretsen}{\ge} 0 \therefore \text{ in order to prove } (2),$$

$$\text{it suffices to prove : LHS of } ② \overset{?}{\geq} (1174R - 377r)(s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow (16200R^2 - 11030Rr + 2089r^2)s^4 - \begin{pmatrix} 64(7027R^3 - 6824R^2r + 2222Rr^2 - 236r^3) - \\ 16R^3 + 4R^2r - 8Rr^2 + 30r^3 \end{pmatrix} s^2 +$$

$$r^2 \begin{pmatrix} 128(23814R^4 - 30000R^3r + 14164R^2r^2 - 3012Rr^3 + 236r^4) - \\ 32R^3r + 72R^2r^2 - 2Rr^3 - 7r^4 \end{pmatrix} \boxed{\overset{?}{\underset{③}{\geq}}} 0 \text{ and } \because$$

$$(16200R^2 - 11030Rr + 2089r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{ in order to prove } ③, \text{ it suffices to prove :}$$

$$\text{LHS of } ③ \overset{?}{\geq} (16200R^2 - 11030Rr + 2089r^2)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (17172R^3 - 19557R^2r + 8737Rr^2 - 1454r^3)s^2 \boxed{\overset{?}{\underset{④}{\geq}}}$$

$$r \begin{pmatrix} 64(4293R^4 - 6155R^3r + 3482R^2r^2 - 877Rr^3 + 86r^4) + \\ 8R^3r + 32R^2r^2 - 15Rr^3 + 2r^4 \end{pmatrix}$$

$$\text{Now, } (17172R^3 - 19557R^2r + 8737Rr^2 - 1454r^3)s^2 \overset{\text{Rouche + Euler}}{\geq}$$

$$\left(16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r}\right) \overset{?}{\geq} \text{RHS of } ④$$

$$\Leftrightarrow 6156t^4 - 17172t^3 + 11159t^2 - 3164t + 572 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(6156t^3 - 4860t^2 + 1439t - 286) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow ④ \Rightarrow ③ \Rightarrow$$

$$② \Rightarrow ① \text{ is true } \therefore 3\sum_{\text{cyc}} m_a w_a \leq \sqrt{(9s^2 - 80Rr - 2r^2)(4s^2 - 28Rr + 29r^2)}$$

$$\overset{\substack{\text{Mohamed Amine Ben Ajiba} \\ \text{and} \\ \text{Chu-Yang}}}{\leq} \left(\sum_{\text{cyc}} n_a\right)\left(\sum_{\text{cyc}} m_a\right)$$

(Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba – 74; published at www.ssmrmh.ro) *and so,*

$$(n_a + n_b + n_c)(m_a + m_b + m_c) \geq 3(m_a w_a + m_b w_b + m_c w_c) \ \forall \ ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)