ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC following relationship holds :

$$|m_b - m_c| \le \sqrt{\frac{1}{2}(s^2 + 2Rr - 31r^2)}$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$ $c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$, $y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2), y^5 + z^5 =$ $x^5\left(m((m^2-2n)^2 + n^2 - nm^2)\right), y^6 + z^6 = x^6((m^3 - 3nn)^2 - 2n^3)$, and now,

$$|m_b - m_c| \overset{?}{\le} \sqrt{\frac{1}{2}(s^2 + 2Rr - 31r^2)} \Leftrightarrow m_b^2 + m_c^2 - \frac{s^2 + 2Rr - 31r^2}{2} \overset{?}{\le} 2m_b m_c$$

and it's trivially true when : $m_b^2 + m_b^2 - \frac{s^2 + 2Rr - 31r^2}{2} \le 0$ and now we focus on $m_b^2 + m_b^2 - \frac{s^2 + 21Rr - 31r^2}{2} > 0$ and then, it suffices to prove :

$$4m_b^2 m_c^2 \overset{?}{\ge} \left(m_b^2 + m_c^2 - \frac{s^2 + 2Rr - 31r^2}{2}\right)^2$$

$$\Leftrightarrow \frac{1}{4}\left(\sum_{cyc} x\right)^2 \cdot \left(4y\left(\sum_{cyc} x\right) + (z - x)^2\right)\left(4z\left(\sum_{cyc} x\right) + (x - y)^2\right) \overset{?}{\ge}$$

$$\frac{1}{16}\left(\begin{matrix}\left(\sum_{cyc} x\right)\left(4(y+z)\left(\sum_{cyc} x\right) + (z-x)^2 + (x-y)^2\right) - \\ 2\left(\sum_{cyc} x\right)^2 - \prod_{cyc}(y+z) + 62xyz\end{matrix}\right)^2$$

$$\Leftrightarrow 4x^6 + 16x^5(y+z) - x^4(y^2+z^2) + 134x^4yz - 32x^3(y^3+z^3) + 672x^3.yz(y+z) - 10x^2(y^4+z^4) + 366x^2.yz(y^2+z^2) - 2844x^2y^2z^2 + 16x(y^5+z^5) - 112x.yz(y^3+z^3) - 48x.y^2z^2(y+z) + 7(y^6+z^6) + 76yz(y^4+z^4) + 236y^2z^2(y^2+z^2) + 334y^3z^3 \boxed{\overset{?}{\ge}} 0$$ (1) and via set of relations "S",

in order to prove (1), it suffices to prove, following simplification :

$$\overbrace{(5m-58)(m-62)}^{\Omega_1}n^2 - \overbrace{(34m^4 - 192m^3 + 406m^2 + 768m + 136)}^{\Omega_2}n -$$

$$\overbrace{(m-1)^2(7m^4+30m^3+43m^2+24m+4)}^{\Omega_3}\ \boxed{\underset{②}{\overset{?}{\leq}}}\ 0$$

Now, $\Omega_2 = 34m^4 - 192m^3 + 406m^2 + 768m + 136 =$

$$\frac{(1088m^2 - 7504m + 12939)(4m^2+5m) + 37732m^2 + 33609m + 17408}{128} > 0$$

$(\because$ discriminant of $(1088m^2 - 7504m + 12939) = -512 < 0)$ $\therefore$ if $\frac{58}{5} \leq m \leq 62$,

then : ② is trivially true and so, we now consider the cases when :

$\boxed{m \in \left(0, \frac{58}{5}\right) \text{ or } m \in (62, \infty)}$ and now, LHS of ② is a quadratic polynomial in "n" with

discriminant, $\boxed{\delta} = \Omega_2^2 + 4\Omega_1\Omega_3$

$$= 144(m+1)^2\begin{pmatrix}((3m^2-31)^2 + 26m^3 + 39m^2 + 656m + 480)(m-1)^2 + \\ 2256m + 48\end{pmatrix} > 0,$$

hence, since $n \overset{AM-GM}{\leq} \frac{m^2}{4}$ $\therefore$ in order to prove ②, it suffices to prove :

$$\Omega_1.\frac{m^2}{2} \overset{?}{\leq} \Omega_2 + \sqrt{\delta} \Leftrightarrow \Omega_1.m^2 - 2\Omega_2\ \boxed{\underset{(m)}{\overset{?}{\leq}}}\ 2\sqrt{\delta} \text{ AND } \Omega_1.2n\ \boxed{\underset{(n)}{\overset{?}{\geq}}}\ \Omega_2 - \sqrt{\delta}$$

We note that (m) is trivially true if : $\Omega_1.m^2 - 2\Omega_2 \leq 0$ and so, we now focus on the case when : $\Omega_1.m^2 - 2\Omega_2 > 0$ and then :

$$63m^4 - 16m^3 - 2784m^2 + 1536m + 272 < 0$$

$$\Rightarrow (m-7)(63m^3 + 425m^2 + 191m + 2873) + 20383 < 0$$

$$\Rightarrow (m-7)(63m^3 + 425m^2 + 191m + 2873) < -20383 < 0 \Rightarrow \boxed{m < 7} \to \text{(i)}$$

and now, $\Omega_1.m^2 - 2\Omega_2 > 0 \Rightarrow$ in order to prove (m), it suffices to prove :

$$\left(-(63m^4 - 16m^3 - 2784m^2 + 1536m + 272)\right)^2 \overset{?}{\leq}$$

$$576(m+1)^2\left(\begin{pmatrix}(3m^2-31)^2 + 26m^3 + 39m^2 + \\ 656m + 480\end{pmatrix}(m-1)^2 + 2256m + 48\right)$$

$$\Leftrightarrow (m-2)^2((m-7)-55)\begin{pmatrix}1215m^4(m-7) - 1449m^4 - 46984m^3 - \\ 11664m^2 - 4560m - 928\end{pmatrix} \overset{?}{\geq} 0$$

$\to$ true via (i) $\Rightarrow \boxed{\text{(m) is true}}$; Again, $\Omega_1.2n + \sqrt{\delta} > \sqrt{\delta}\ (\because > 0) = \sqrt{\Omega_2^2 + 4\Omega_1\Omega_3}$

$> \sqrt{\Omega_2^2}\ (\because \Omega_3 > 0) = \Omega_2\ (\because \Omega_2 > 0$, deduced earlier$) \Rightarrow \boxed{\text{(n) is true}}$ $\therefore$ both

(m) and (n) are true $\Rightarrow$ ② $\Rightarrow$ ① is true $\therefore$ $|m_b - m_c| \leq \sqrt{\frac{1}{2}(s^2 + 2Rr - 31r^2)}$

$\forall\, \Delta\, ABC$, " = " iff $\Delta\, ABC$ is equilateral (QED)