ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{g_a n_a} \le \sqrt{\frac{7R}{6r} - \frac{4}{3}} . h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$
$c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$
and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$,
$y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2)$,
$y^5 + z^5 = x^5\left(m((m^2 - 2n)^2 + n^2 - nm^2)\right)$, and now, $\sqrt{g_a n_a} \overset{?}{\le} \sqrt{\frac{7R}{6r} - \frac{4}{3}} . h_a$

$$\overset{\text{Bogdan Fustei}}{\Leftrightarrow} s^2(s-a)^2 + \frac{4s(s-a)(s-b)(s-c)(b-c)^2}{a^2} \overset{?}{\le}$$

$$\left(\frac{7}{6} . \frac{(y+z)(z+x)(x+y)}{4xyz} - \frac{4}{3}\right)^2 . \frac{16s^2(s-a)^2(s-b)^2(s-c)^2}{a^4} \Leftrightarrow$$

$$\frac{(y+z)^2 . x(x+y+z) + 4yz(y-z)^2}{(y+z)^2} \overset{?}{\le} \frac{(7(y+z)(z+x)(x+y) - 32xyz)^2}{36x^2y^2z^2} . \frac{x(x+y+z) . y^2z^2}{(y+z)^4}$$

$$\Leftrightarrow 49x^5(y+z)^2 + 147x^4(y^3+z^3) - 7x^4yz(y+z) + 111x^3(y^4+z^4) - 354x^3 . yz(y^2+z^2) + 94x^3 . y^2z^2 + 13x^2(y^5+z^5) - 187x^2 . yz(y^3+z^3) - 50x^2 . y^2z^2(y+z) - 46x . yz(y^4+z^4) - 7x . y^2z^2(y^2+z^2) + 78x . y^3z^3 + 49y^2z^2(y^3+z^3) \boxed{\overset{?}{\underset{(1)}{\ge}}} 0$$

and via set of relations "S",

in order to prove (1), it suffices to prove, following simplification :

$$\overbrace{(49m^3 + 177m^2 + 576m + 1024)}^{\Omega_1} n^2 - \overbrace{(46m^4 + 252m^3 + 798m^2 + 448m)}^{\Omega_2} n + \overbrace{(13m^5 + 111m^4 + 147m^3 + 49m^2)}^{\Omega_3} \boxed{\overset{?}{\underset{(2)}{\ge}}} 0$$

Now, LHS of (2) is a quadratic polynomial in "n" with discriminant,
$\boxed{\delta} = \Omega_2^2 - 4\Omega_1\Omega_3 = 432m^4(-m^4 - 18m^3 - m^2 + 48m + 80)$ and when :
$\delta \le 0$, then : (2) is trivially true and when : $\delta > 0$,
then : $m^4 + 18m^3 + m^2 - 48m - 80 < 0$

$$\Rightarrow \frac{(729m^3 + 14661m^2 + 31680m + 31888)(9m - 19)}{6561} < -\frac{80992}{6561} < 0$$

$$\Rightarrow m < \frac{19}{9} \to (m)$$

$$\text{Now, since } n \overset{AM-GM}{\le} \frac{m^2}{4} \therefore \text{ in order to prove } ②, \text{ it suffices to prove :}$$

$$\Omega_1 . \frac{m^2}{2} \overset{?}{\le} \Omega_2 - \sqrt{\delta} \Leftrightarrow 2\Omega_2 - \Omega_1 m^2 \overset{?}{\ge} 2\sqrt{\delta}$$

$$\Leftrightarrow m(-49m^4 - 85m^3 - 72m^2 + 572m + 896) \boxed{\overset{?}{\underset{③}{\ge}}} 2\sqrt{\delta}$$

$$\text{Now, } 49m^4 + 85m^3 + 72m^2 - 572m - 896 =$$

$$\frac{49(729m^3 + 2754m^2 + 6984m + 5948)(9m - 19)}{6561} +$$

$$\frac{(2430m^2 + 288m + 44)(9m - 19)}{6561} - 51 - \frac{5621}{6561} \overset{via\,(m)}{<} 0$$

$$\Rightarrow m(-49m^4 - 85m^3 - 72m^2 + 572m + 896) > 0 \text{ and so,}$$

$$\text{in order to prove } ③, \text{ it suffices to prove :}$$

$$m^2(-49m^4 - 85m^3 - 72m^2 + 572m + 896)^2 \overset{?}{\ge}$$

$$1728m^4(-m^4 - 18m^3 - m^2 + 48m + 80) \Leftrightarrow$$

$$7m^2(m+1)(m-2)^2 \begin{pmatrix} 343m^5 + 2219m^4 + 8944m^3 + 23644m^2 + \\ 36608m + 28672 \end{pmatrix} \overset{?}{\ge} 0$$

$$\to \text{true} \because m > 0 \Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true} \therefore \sqrt{g_a n_a} \le \sqrt{\frac{7R}{6r} - \frac{4}{3}} . h_a \ \forall \, \Delta\, ABC,$$

$$'' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$