ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{w_a n_a} \le \sqrt{\frac{2R}{r} - 3} \cdot h_a$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$ $c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$, $y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2)$,

$y^5 + z^5 = x^5\left(m((m^2 - 2n)^2 + n^2 - nm^2)\right)$, and now, $\sqrt{w_a n_a} \overset{?}{\le} \sqrt{\frac{2R}{r} - 3}.h_a$

$$\overset{\text{Bogdan Fustei}}{\Leftrightarrow} \frac{4bc}{(b+c)^2}.s(s-a)\left(s(s-a) + \frac{s}{a}.(b-c)^2\right) \overset{?}{\le}$$

$$\left(\frac{(y+z)(z+x)(x+y)}{2xyz} - 3\right)^2.\frac{16s^2(s-a)^2(s-b)^2(s-c)^2}{a^4}$$

$$\Leftrightarrow \left((y+z)(z+x)(x+y) - 6xyz\right)^2(2x+y+z)^2 \overset{?}{\ge}$$
$$x(y+z)^3(z+x)(x+y)(x(y+z) + (y-z)^2)$$

$$\Leftrightarrow 4x^6(y+z)^2 + 12x^5(y^3+z^3) - 12x^5.yz(y+z) + 12x^4(y^4+z^4) -$$
$$40x^4.yz(y^2+z^2) + 40x^4.y^2z^2 + 4x^3(y^5+z^5) - 20x^3.yz(y^3+z^3) +$$
$$16x^3.y^2z^2(y+z) + x^2.yz(y^4+z^4) - 4x^2.y^2z^2(y^2+z^2) - 10x^2.y^3z^3 +$$
$$xyz(y^5+z^5) + x.y^2z^2(y^3+z^3) - 2x.y^3z^3(y+z) + y^2z^2(y+z)^4 \boxed{\overset{?}{\underset{(1)}{\ge}}} 0$$

& via set of relations "S", in order to prove (1), it suffices to prove, following simplification :

$$\overbrace{(m^4 - 4m^3 - 8m^2 + 96m + 144)}^{\Omega_1}n^2 + \overbrace{(m^5 + m^4 - 40m^3 - 88m^2 - 48m)}^{\Omega_2}n +$$
$$\overbrace{(4m^5 + 12m^4 + 12m^3 + 4m^2)}^{\Omega_3} \boxed{\overset{?}{\underset{(2)}{\ge}}} 0;$$

We have : $m^4 - 4m^3 - 8m^2 + 96m + 144 =$

$$\frac{(2m-9)^2(2m+5)^2}{16} + \frac{168m^2 + 816m + 339}{16} > 0 \therefore (2) \text{ is trivially true when :}$$

$m^5 + m^4 - 40m^3 - 88m^2 - 48m \ge 0$ & when : $m^5 + m^4 - 40m^3 - 88m^2 - 48m$ < 0, then : $(m-7)(m^3 + 8m^2 + 16m + 24) + 120 < 0$

$\Rightarrow (m-7)(m^3 + 8m^2 + 16m + 24) < -120 < 0 \Rightarrow m < 7 \to (m)$ and $\because$ LHS of (2) is a quadratic polynomial in "n" with discriminant,

$$\boxed{\delta} = \Omega_2^2 - 4\Omega_1\Omega_3 = m^4(m+1)^2(m^4 - 16m^3 - 32m^2 + 96m + 192) \therefore \text{if } \delta \le 0,$$

then : ② is trivially true and when : $\delta > 0$, then :

$$m^4 - 16m^3 - 32m^2 + 96m + 192 > 0 \Rightarrow$$

$$(m^3 - 13m^2 - 71m - 117)(m - 3) - 159 > 0 \Rightarrow$$

$$(m^3 - 13m^2 - 71m - 117)(m - 3) > 159 > 0 \text{ and } \because m^3 - 13m^2 - 71m - 117 =$$

$$m^2(m - 7) - 6m^2 - 71m - 117 \overset{\text{via (m)}}{<} 0 \therefore m < 3 \to \text{(n)}$$

Now, since $n \overset{\text{AM-GM}}{\le} \dfrac{m^2}{4}$ $\therefore$ in order to prove ②, it suffices to prove :

$$\Omega_1 . \frac{m^2}{2} \overset{?}{\le} -\Omega_2 - \sqrt{\delta} \Leftrightarrow -2\Omega_2 - \Omega_1 m^2 \overset{?}{\ge} 2\sqrt{\delta}$$

$$\Leftrightarrow m\left(-m^5 + 2m^4 + 6m^3 - 16m^2 + 32m + 96\right) \boxed{\underset{③}{\overset{?}{\ge}}} 2\sqrt{\delta}$$

We have : $m^5 - 2m^4 - 6m^3 + 16m^2 - 32m - 96 =$

$$m^4(m - 3) + m^3(m - 3 - 3) + 16m(m - 3) + 16(m - 3 - 3) \overset{\text{via (n)}}{<} 0$$

$$\Rightarrow m\left(-m^5 + 2m^4 + 6m^3 - 16m^2 + 32m + 96\right) > 0 \text{ and so,}$$

in order to prove ③, it suffices to prove :

$$m^2\left(-m^5 + 2m^4 + 6m^3 - 16m^2 + 32m + 96\right)^2 \overset{?}{\ge}$$

$$4m^4(m+1)^2(m^4 - 16m^3 - 32m^2 + 96m + 192) \Leftrightarrow$$

$$m^2(m+2)^2(m-2)^2\left((2m+5)^2\left(\frac{(4m^2 - 18m)^2 + 232m^2 - 600m + 1317}{64}\right) + \frac{13236m + 3939}{64}\right)$$

$$\overset{?}{\ge} 0 \to \text{true} \because m > 0 \text{ and } \because \text{discriminant of } 232m^2 - 600m + 1317 < 0$$

$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true} \therefore \sqrt{w_a n_a} \le \sqrt{\frac{2R}{r} - 3} . h_a \ \forall \ \Delta\, ABC,$$

$'' = ''$ iff Δ ABC is equilateral (QED)