ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$m_a^2 - w_a^2 \leq \sqrt{\frac{2}{27}}\left|b^2 - c^2\right|$$

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$$m_a^2 = s(s-a) + \frac{1}{4}(b-c)^2, w_a^2 = s(s-a) - \frac{s(s-a)}{(b+c)^2}(b-c)^2$$

$$m_a^2 - w_a^2 = \frac{(b-c)^2\left((b+c)^2 + 4s(s-a)\right)}{4(b+c)^2} = \frac{(b-c)^2(2(b+c)^2 - a^2)}{4(b+c)^2}$$

$$let\ x = b+c, y = |b-c|\ then\ \left|b^2 - c^2\right| = xy, \quad m_a^2 - w_a^2 = \frac{y^2(2x^2 - a^2)}{4x^2}$$

$$now, \frac{m_a^2 - w_a^2}{|b^2 - c^2|} = \frac{y(2x^2 - a^2)}{4x^3} = \frac{\frac{y}{x}\left(2 - \frac{a^2}{x^2}\right)}{4} = \frac{t(2-u^2)}{4} \overset{t<u}{\leq} \frac{t(2-t^2)}{4}$$

$$\left(t = \frac{y}{x} = \frac{|b-c|}{b+c}, u = \frac{a}{b+c}\ and\ clearly, 0 < t < u < 1\ as\ |b-c| < a\right)$$

$$let\ f(t) = \frac{t(2-t^2)}{4}\ then\ f'(t) = \frac{2-3t^2}{4}, f''(t) = -\frac{3t}{2}$$

$$now\ f'(t) = 0 \Rightarrow t = \sqrt{\frac{2}{3}}\ \&\ f''(t)\ at\ t = \sqrt{\frac{2}{3}} = -\frac{3}{2}\sqrt{\frac{2}{3}} < 0\ so$$

$$f_{Max} = \frac{\sqrt{\frac{2}{3}}\left(2 - \frac{2}{3}\right)}{4} = \sqrt{\frac{2}{27}}$$

$$\frac{m_a^2 - w_a^2}{|b^2 - c^2|} = \frac{y(2x^2 - a^2)}{4x^3} = \frac{\frac{y}{x}\left(2 - \frac{a^2}{x^2}\right)}{4} = \frac{t(2-u^2)}{4} \overset{t<u}{\leq} \frac{t(2-t^2)}{4} = f(t) \leq \sqrt{\frac{2}{27}}$$

$$m_a^2 - w_a^2 \leq \sqrt{\frac{2}{27}}\left|b^2 - c^2\right|$$

Equality holds for an equilateral triangle.