ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$min\{m_a,m_b,m_c\}\ \le \frac{1}{2}\sqrt{s^2-4Rr+17r^2}\le \sqrt{s^2-7Rr-4r^2}\le max\{m_a,m_b,m_c\}$$

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$W.L.O.G.$ $let's\ assume\ that$ $a\le b\le c$. $Then\ we\ have$: $m_a\ge m_b\ge m_c$

$$min\{m_a,m_b,m_c\}=m_c\ and\ max\{m_a,m_b,m_c\}=m_a$$

$Also\ let$ $(a,b,c)=(y+z,z+x,x+y)$ $which\ gives$ $x\ge y\ge z$ $with$ $x,y,z>0$.

$$Here, (s^2-16Rr+5r^2)+8r(R-2r)\ \overbrace{\ge}^{Gerretsen+Euler}\ 0\ \Leftrightarrow\ s^2-8Rr-11r^2\ge 0$$

$$\Leftrightarrow 4(s^2-7Rr-4r^2)-(s^2-4Rr+17r^2)\ge 0\Leftrightarrow \frac{1}{4}(s^2-4Rr+17r^2)\le\ s^2-7Rr-4r^2$$

$$\therefore \frac{1}{2}\sqrt{s^2-4Rr+17r^2}\le \sqrt{s^2-7Rr-4r^2}$$

$i.)$ $Let's\ prove\ that$: $min\{m_a,m_b,m_c\}=m_c\le\frac{1}{2}\sqrt{s^2-4Rr+17r^2}$

$Squaring$ $m_c\le\frac{1}{2}\sqrt{s^2-4Rr+17r^2}$ $and\ substituting$ $m_c^2=s^2-r^2-4Rr-\frac{3}{4}c^2$ $yields$:

$$s^2-4Rr-7r^2\le c^2\ \overbrace{\Leftrightarrow}^{s^2-4Rr=\frac{a^2+b^2+c^2}{2}+r^2}\ a^2+b^2-c^2\le 12r^2$$

$\Leftrightarrow z(x+y+z)^2\le xy(x+y+7z)\Leftrightarrow (x-z)(y-z)(x+y+7z)+4z^2[(x-z)+(y-z)]\ge 0$ $which\ is\ true\ because$ $x\ge z$.

$ii.)$ $Let's\ prove\ that$: $\sqrt{s^2-7Rr-4r^2}\le max\{m_a,m_b,m_c\}=m_a$

$Squaring$ $\sqrt{s^2-7Rr-4r^2}\le m_a$ $and\ substituting$ $m_a^2=s^2-r^2-4Rr-\frac{3}{4}a^2$ $yields$:

$$s^2-7Rr-4r^2\le s^2-r^2-4Rr-\frac{3}{4}a^2\quad\Leftrightarrow\quad a^2\le 4Rr+4r^2$$

$$\Leftrightarrow\ (y+z)^2\le\frac{(x+y)(y+z)(z+x)+4xyz}{x+y+z}\Leftrightarrow x^2+\frac{4xyz}{y+z}\ge y^2+yz+z^2.\ Also,$$

$x^2+\frac{4xyz}{y+z}\ \overbrace{\ge}^{x\ge y}\ y^2+\frac{4y^2z}{y+z}$. $Thus\ it\ suffices\ to\ prove\ that$: $y^2+\frac{4y^2z}{y+z}\ge y^2+yz+z^2$

$\Leftrightarrow\frac{4y^2z}{y+z}\ge yz+z^2\Leftrightarrow 4y^2\ge(y+z)^2\Leftrightarrow y\ge z$ $which\ is\ true\ because$ $y\ge z$.

$$min\{m_a,m_b,m_c\}\ \le\frac{1}{2}\sqrt{s^2-4Rr+17r^2}\le\sqrt{s^2-7Rr-4r^2}\le max\{m_a,m_b,m_c\}$$

Equality holds if and only if the triangle is equilateral.