ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$2R-r-2\sqrt{R(R-2r)}\ \le min\{r_a,r_b,r_c\}\le min\{h_a,h_b,h_c\}$$

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$$W.L.O.G.\ let's\ assume\ that\ a\le b\le c. Then\ we\ have:$$

$$min\{r_a,r_b,r_c\}=r_a\quad and\quad min\{h_a,h_b,h_c\}=h_c$$

$$Here, a\le b\Leftrightarrow c\le b+c-a\Leftrightarrow c\le 2s-2a\Leftrightarrow \frac{\Delta}{s-a}\le\frac{2\Delta}{c}$$

$$\Leftrightarrow r_a\le h_c\quad \therefore min\{r_a,r_b,r_c\}\le min\{h_a,h_b,h_c\}$$

$$Now,\qquad (r_b+r_c)\left(\frac{1}{r_b}+\frac{1}{r_c}\right)\overbrace{\ge}^{AM-HM}4$$

$$Via\ identities\ \frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\ \ and\ \ r_a+r_b+r_c=4R+r, we\ substitute:$$

$$(4R+r-r_a)\left(\frac{1}{r}-\frac{1}{r_a}\right)\ge 4\Leftrightarrow (4R+r-r_a)(r_a-r)\ge 4rr_a$$

$$\Leftrightarrow r_a^2-2(2R-r)r_a+r(4R+r)\le 0$$

$$\Leftrightarrow\left(r_a-2R+r+2\sqrt{R(R-2r)}\right)\left(r_a-2R+r-2\sqrt{R(R-2r)}\right)\le 0$$

$$\therefore 2R-r-2\sqrt{R(R-2r)}\le r_a\le 2R-r+2\sqrt{R(R-2r)}$$

$$Extracting\ the\ lower\ bound\ gives:$$

$$2R-r-2\sqrt{R(R-2r)}\le r_a=min\{r_a,r_b,r_c\}$$

$$Therefore,\qquad 2R-r-2\sqrt{R(R-2r)}\ \le min\{r_a,r_b,r_c\}\le min\{h_a,h_b,h_c\}$$

$$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$$