

ROMANIAN MATHEMATICAL MAGAZINE

In $\triangle ABC$ the following relationship holds:

$$2R - r - 2\sqrt{R(R - 2r)} \leq \min\{r_a, r_b, r_c\} \leq \min\{h_a, h_b, h_c\}$$

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W. L. O. G. let's assume that $a \leq b \leq c$. Then we have:

$$\min\{r_a, r_b, r_c\} = r_a \quad \text{and} \quad \min\{h_a, h_b, h_c\} = h_c$$

$$\text{Here, } a \leq b \Leftrightarrow c \leq b + c - a \Leftrightarrow c \leq 2s - 2a \Leftrightarrow \frac{\Delta}{s - a} \leq \frac{2\Delta}{c}$$

$$\Leftrightarrow r_a \leq h_c \quad \therefore \min\{r_a, r_b, r_c\} \leq \min\{h_a, h_b, h_c\}$$

$$\text{Now, } (r_b + r_c) \left(\frac{1}{r_b} + \frac{1}{r_c} \right) \stackrel{AM-HM}{\geq} 4$$

Via identities $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$ and $r_a + r_b + r_c = 4R + r$, we substitute:

$$(4R + r - r_a) \left(\frac{1}{r} - \frac{1}{r_a} \right) \geq 4 \Leftrightarrow (4R + r - r_a)(r_a - r) \geq 4rr_a$$

$$\Leftrightarrow r_a^2 - 2(2R - r)r_a + r(4R + r) \leq 0$$

$$\Leftrightarrow (r_a - 2R + r + 2\sqrt{R(R - 2r)})(r_a - 2R + r - 2\sqrt{R(R - 2r)}) \leq 0$$

$$\therefore 2R - r - 2\sqrt{R(R - 2r)} \leq r_a \leq 2R - r + 2\sqrt{R(R - 2r)}$$

Extracting the lower bound gives:

$$2R - r - 2\sqrt{R(R - 2r)} \leq r_a = \min\{r_a, r_b, r_c\}$$

$$\text{Therefore, } 2R - r - 2\sqrt{R(R - 2r)} \leq \min\{r_a, r_b, r_c\} \leq \min\{h_a, h_b, h_c\}$$

Equality holds if and only if the triangle is equilateral.