ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\max\{n_a, n_b, n_c\} \le \max\{r_a, r_b, r_c\} \le 2R - r + 2\sqrt{R(R-2r)}$$

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$$r_a + r = \frac{r.p}{p-a} + \frac{r.p}{p} = \frac{r(b+c)}{p-a}\ (p \to \text{semi} - \text{perimeter})$$

$$= \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \therefore r_a + r \overset{①}{=} 4Rsc$$

$$\left(\text{where } c = \cos\frac{B-C}{2}, s = \sin\frac{A}{2}\right) \text{ and again,}$$

$$2R + 2\sqrt{R(R-2r)} \overset{②}{=} 2R + 2R.\sqrt{1 - 4sc + 4s^2}$$

$$\text{Now, ① and ②} \Rightarrow r_a \overset{?}{\le} 2R - r + 2\sqrt{R(R-2r)}$$

$$\Leftrightarrow 4Rsc \overset{?}{\le} 2R + 2R.\sqrt{1-4sc+4s^2} \Leftrightarrow 2sc - 1 \boxed{\underset{(*)}{\overset{?}{\le}}} \sqrt{1-4sc+4s^2} \text{ and if}$$

$2sc - 1 < 0$, *then* : $(*)$ is trivially true and when : $2sc - 1 \ge 0$ and then :

$$(*) \Leftrightarrow 4s^2c^2 - 4sc + 1 \overset{?}{\le} 1 - 4sc + 4s^2 \Leftrightarrow 4s^2(c^2 - 1) \le 0 \to \text{true}$$

$$\because c^2 = \cos^2\frac{B-C}{2} \le 1 \Rightarrow (*) \text{ is true } \forall \Delta \text{ ABC}$$

$$\therefore r_a \le 2R - r + 2\sqrt{R(R-2r)} \text{ and similarly}$$

$$r_b, r_c \le 2R - r + 2\sqrt{R(R-2r)} \text{ and so,}$$

$$\boxed{\boxed{\max\{r_a, r_b, r_c\} \le 2R - r + 2\sqrt{R(R-2r)}}} \text{ and now, } n_a^2 \overset{\text{Bogdan Fustei}}{=}$$

$$s^2 - \frac{r}{R}.s^2.\sec^2\frac{A}{2} \therefore \text{if } a \le b, \text{then, as } \sec^2\frac{A}{2} \le \sec^2\frac{B}{2}, \text{hence } n_a^2 \ge n_b^2 \to ①$$

and we seggregate into the following cases :

$\boxed{\text{Case 1}}$ $a \le b \le c$ and then : $\max\{n_a, n_b, n_c\} \overset{?}{\le} \max\{r_a, r_b, r_c\} \overset{\text{via ①}}{\Leftrightarrow} n_a^2 \overset{?}{\le} r_c^2$

$$\overset{\text{Bogdan Fustei}}{\Leftrightarrow} s(s-a) + \frac{s}{a}(b-c)^2 \overset{?}{\le} \frac{s(s-a)(s-b)(s-c)}{(s-c)^2}$$

$$\Leftrightarrow \frac{xy}{z} \overset{?}{\ge} x + \frac{(y-z)^2}{y+z}\ (x = s-a, y = s-b, z = s-c)$$

$$\Leftrightarrow (y-z)\left(xy + z^2 + z(x-y)\right) \overset{?}{\ge} 0 \to \text{true} \because a \le b \le c \Rightarrow x \ge y \ge z$$

$\boxed{\text{Case 2}}$ $a \le c \le b$ and then : $\max\{n_a, n_b, n_c\} \overset{?}{\le} \max\{r_a, r_b, r_c\} \overset{\text{via ①}}{\Leftrightarrow} n_a^2 \overset{?}{\le} r_b^2$

$$\Leftrightarrow (z-y)\left(xz + y^2 + y(x-z)\right) \overset{?}{\ge} 0 \to \text{true} \because a \le c \le b \Rightarrow x \ge z \ge y$$

$\boxed{\text{Case 3}}$ $b \leq c \leq a$ and then : $\max\{n_a, n_b, n_c\} \overset{?}{\leq} \max\{r_a, r_b, r_c\} \overset{\text{via } ①}{\Leftrightarrow} n_b^2 \overset{?}{\leq} r_a^2$

$\Leftrightarrow (z - x)\left(yz + x^2 + x(y - z)\right) \overset{?}{\geq} 0 \to$ true $\because b \leq c \leq a \Rightarrow y \geq z \geq x$

$\boxed{\text{Case 4}}$ $b \leq a \leq c$ and then : $\max\{n_a, n_b, n_c\} \overset{?}{\leq} \max\{r_a, r_b, r_c\} \overset{\text{via } ①}{\Leftrightarrow} n_b^2 \overset{?}{\leq} r_c^2$

$\Leftrightarrow (x - z)\left(xy + z^2 + z(y - x)\right) \overset{?}{\geq} 0 \to$ true $\because b \leq a \leq c \Rightarrow y \geq x \geq z$

$\boxed{\text{Case 5}}$ $c \leq a \leq b$ and then : $\max\{n_a, n_b, n_c\} \overset{?}{\leq} \max\{r_a, r_b, r_c\} \overset{\text{via } ①}{\Leftrightarrow} n_c^2 \overset{?}{\leq} r_b^2$

$\Leftrightarrow (x - y)\left(zx + y^2 + y(z - x)\right) \overset{?}{\geq} 0 \to$ true $\because c \leq a \leq b \Rightarrow z \geq x \geq y$

$\boxed{\text{Case 6}}$ $c \leq b \leq a$ and then : $\max\{n_a, n_b, n_c\} \overset{?}{\leq} \max\{r_a, r_b, r_c\} \overset{\text{via } ①}{\Leftrightarrow} n_c^2 \overset{?}{\leq} r_a^2$

$\Leftrightarrow (y - x)\left(yz + x^2 + x(z - y)\right) \overset{?}{\geq} 0 \to$ true $\because c \leq b \leq a \Rightarrow z \geq y \geq x$

$\therefore$ combining *all* cases, $\boxed{\boxed{\max\{n_a, n_b, n_c\} \leq \max\{r_a, r_b, r_c\}}}$ *and* so,

$\max\{n_a, n_b, n_c\} \leq \max\{r_a, r_b, r_c\} \leq 2R - r + 2\sqrt{R(R - 2r)}\ \forall\, \Delta\, ABC,$
$'' = ''$ iff Δ ABC is isosceles (QED)