ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$2R - r - 2\sqrt{R(R-2r)} \le \min\{r_a, r_b, r_c\} \le \min\{h_a, h_b, h_c\}$$

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$$r_a + r = \frac{r.p}{p-a} + \frac{r.p}{p} = \frac{r(b+c)}{p-a} \ (p \to \text{semi} - \text{perimeter})$$

$$= \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \therefore r_a + r \overset{(1)}{=} 4Rsc\left(c = \cos\frac{B-C}{2}, s = \sin\frac{A}{2}\right)$$

$$\text{and again}, 2R - 2\sqrt{R(R-2r)} \overset{(2)}{=} 2R - 2R.\sqrt{1 - 4sc + 4s^2}$$

$$\text{Now}, (1) \text{ and } (2) \Rightarrow r_a \overset{?}{\ge} 2R - r - 2\sqrt{R(R-2r)}$$

$$\Leftrightarrow 4Rsc \overset{?}{\ge} 2R - 2R.\sqrt{1-4sc+4s^2} \Leftrightarrow \sqrt{1-4sc+4s^2} \boxed{\overset{?}{\underset{(*)}{\ge}}} 1 - 2sc \text{ and if}$$

$$1 - 2sc < 0, \text{then} : (**) \text{ is trivially true and when} : 1 - 2sc \ge 0, \text{then} :$$

$$(*) \Leftrightarrow 1 - 4sc + 4s^2 \overset{?}{\ge} 4s^2c^2 - 4sc + 1 \Leftrightarrow 4s^2(1-c^2) \overset{?}{\ge} 0 \to \text{true}$$

$$\because c^2 = \cos^2\frac{B-C}{2} \le 1 \Rightarrow (*) \text{ is true } \forall \Delta \text{ ABC}$$

$$\therefore r_a \ge 2R - r - 2\sqrt{R(R-2r)} \text{ and similarly } r_b, r_c \ge 2R - r - 2\sqrt{R(R-2r)}$$

$$\text{and so}, \boxed{\boxed{\min\{r_a, r_b, r_c\} \ge 2R - r - 2\sqrt{R(R-2r)}}} \text{ and now},$$

$$\text{we seggregate into the following cases} :$$

$$\boxed{\text{Case 1}}\ a \le b \le c \text{ and then} : \min\{r_a, r_b, r_c\} \overset{?}{\le} \min\{h_a, h_b, h_c\} \Leftrightarrow r_a \overset{?}{\le} h_c$$

$$\Leftrightarrow b + c - a \overset{?}{\ge} c \Leftrightarrow b \overset{?}{\ge} a \to \text{true}$$

$$\boxed{\text{Case 2}}\ a \le c \le b \text{ and then} : \min\{r_a, r_b, r_c\} \overset{?}{\le} \min\{h_a, h_b, h_c\} \Leftrightarrow r_a \overset{?}{\le} h_b$$

$$\Leftrightarrow b + c - a \overset{?}{\ge} b \Leftrightarrow c \overset{?}{\ge} a \to \text{true}$$

$$\boxed{\text{Case 3}}\ b \le c \le a \text{ and then} : \min\{r_a, r_b, r_c\} \overset{?}{\le} \min\{h_a, h_b, h_c\} \Leftrightarrow r_b \overset{?}{\le} h_a$$

$$\Leftrightarrow c + a - b \overset{?}{\ge} a \Leftrightarrow c \overset{?}{\ge} b \to \text{true}$$

$$\boxed{\text{Case 4}}\ b \le a \le c \text{ and then} : \min\{r_a, r_b, r_c\} \overset{?}{\le} \min\{h_a, h_b, h_c\} \Leftrightarrow r_b \overset{?}{\le} h_c$$

$$\Leftrightarrow c + a - b \overset{?}{\ge} c \Leftrightarrow a \overset{?}{\ge} b \to \text{true}$$

$$\boxed{\text{Case 5}}\ c \le a \le b \text{ and then} : \min\{r_a, r_b, r_c\} \overset{?}{\le} \min\{h_a, h_b, h_c\} \Leftrightarrow r_c \overset{?}{\le} h_b$$

$$\Leftrightarrow a + b - c \overset{?}{\ge} b \Leftrightarrow a \overset{?}{\ge} c \to \text{true}$$

$$\boxed{\text{Case 6}}\ c \le b \le a \text{ and then} : \min\{r_a, r_b, r_c\} \overset{?}{\le} \min\{h_a, h_b, h_c\} \Leftrightarrow r_c \overset{?}{\le} h_a$$

$$\Leftrightarrow a + \mathrm{b} - \mathrm{c} \overset{?}{\geq} a \Leftrightarrow \mathrm{b} \overset{?}{\geq} \mathrm{c} \to \text{true} \therefore \text{combining } all\ cases,$$

$$\boxed{\boxed{\min\{\mathrm{r}_a, \mathrm{r}_\mathrm{b}, \mathrm{r}_\mathrm{c}\} \leq \min\{\mathrm{h}_a, \mathrm{h}_\mathrm{b}, \mathrm{h}_\mathrm{c}\}}}\ \ a\text{nd so},$$

$$2\mathrm{R} - \mathrm{r} - 2\sqrt{\mathrm{R}(\mathrm{R} - 2\mathrm{r})} \leq \min\{\mathrm{r}_a, \mathrm{r}_\mathrm{b}, \mathrm{r}_\mathrm{c}\} \leq \min\{\mathrm{h}_a, \mathrm{h}_\mathrm{b}, \mathrm{h}_\mathrm{c}\} \ \forall\, \Delta\, \mathrm{ABC},$$

$$'' = ''\ \text{iff } \Delta\, \mathrm{ABC} \text{ is isosce}l\text{es (QED)}$$