In any Δ ABC with $a \le b \le c$, the following relationships holds :

$$\frac{\sqrt{3}}{2}a \le g_b \le p_b \le \frac{\sqrt{3}}{2}c\ ;\ and\ p_c \le \frac{\sqrt{3}}{2}b \le g_a$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$ $c = x + y$ and $s = x + y + z$ and also, $a \le b \le c \Rightarrow y + z \le z + x \le x + y$

$$\Rightarrow \boxed{x \ge y \ge z} \text{ and now, } \frac{3}{4}a^2 \overset{?}{\le} g_b^2 \overset{\text{via Bogdan Fustei}}{\Leftrightarrow} \frac{3}{4}(y+z)^2 \overset{?}{\le} y^2 + \frac{4xyz}{z+x}$$

$$\Leftrightarrow xy^2 + 10xyz + y^2z \overset{?}{\ge} 3xz^2 + 6yz^2 + 3z^3$$

$$\Leftrightarrow 6yz(x-z) + 3xz(y-z) + z(xy - z^2) \overset{?}{\ge} 0 \to \text{true} \because x \ge y \ge z \therefore \boxed{\frac{\sqrt{3}}{2}a \le g_b}$$

$$\text{Also, } p_b^2 \overset{?}{\le} \frac{3}{4}c^2 \overset{\substack{\text{via Bogdan Fustei}\\ \text{and}\\ \text{Mohamed Amine Ajiba}}}{\Leftrightarrow} s(s-b) + \frac{s(3s+b)(c-a)^2}{(2s+b)^2} \overset{?}{\le} \frac{3}{4}c^2$$

$$\Leftrightarrow (x+y+z)\left(y + \frac{(3(x+y+z)+z+x)(z-x)^2}{(2(x+y+z)+z+x)^2}\right) \overset{?}{\le} \frac{3}{4}(x+y)^2$$

$$\Leftrightarrow 11x^4 + 26x^3y + 54x^3z + 15x^2y^2 + 64x^2yz + 59x^2z^2 \overset{?}{\ge}$$
$$4xy^3 + 18xy^2z + 26xyz^2 + 4y^4 + 28y^3z + 69y^2z^2 + 64yz^3 + 16z^4$$
$$\Leftrightarrow 18xyz(x-y) + 28yz(x^2-y^2) + 18xyz(x-z) + 59z^2(x^2-y^2) +$$
$$54z(x^3 - yz^2) + 8xy(x^2-z^2) + 4xy(x^2-y^2) + 4y(x^3-y^3) + 10y(x^3-yz^2) +$$
$$10y(x^2y - z^3) + 11(x^4 - z^4) + 5(x^2y^2 - z^4) \overset{?}{\ge} 0 \to \text{true} \because x \ge y \ge z$$

$$\therefore \boxed{p_b \le \frac{\sqrt{3}}{2}c}$$

$$\text{Furthermore, } p_c^2 \overset{?}{\le} \frac{3}{4}b^2 \overset{\substack{\text{via Bogdan Fustei}\\ \text{and}\\ \text{Mohamed Amine Ajiba}}}{\Leftrightarrow} s(s-c) + \frac{s(3s+c)(a-b)^2}{(2s+c)^2} \overset{?}{\le} \frac{3}{4}b^2$$

$$\Leftrightarrow (x+y+z)\left(z + \frac{(3(x+y+z)+x+y)(x-y)^2}{(2(x+y+z)+x+y)^2}\right) \overset{?}{\le} \frac{3}{4}(z+x)^2$$

$$\Leftrightarrow 11x^4 + 26x^3z + 54x^3y + 15x^2z^2 + 64x^2yz + 59x^2y^2 \overset{?}{\ge}$$
$$4xz^3 + 18xyz^2 + 26xy^2z + 4z^4 + 28yz^3 + 69y^2z^2 + 64y^3z + 16y^4$$
$$\Leftrightarrow 64yz(x^2-y^2) + 26xz(x^2-y^2) + 59y^2(x^2-z^2) + 28y(x^3-z^3) +$$
$$18xy(x^2-z^2) + 10z^2(x^2-y^2) + 8y(x^3-y^3) + 8(x^4-y^4) + 4z^2(x^2-z^2) +$$

$$xz^2(x-z)+3x(x^3-z^3)\overset{?}{\geq}0 \to \text{true} \because x\geq y\geq z \therefore \boxed{p_c \leq \frac{\sqrt{3}}{2}b} \textit{ and again},$$

$$\frac{3}{4}b^2 \overset{?}{\leq} g_a^2 \overset{\text{via Bogdan Fustei}}{\Leftrightarrow} \frac{3}{4}(z+x)^2 \overset{?}{\leq} x^2+\frac{4xyz}{y+z}$$

$$\Leftrightarrow x^2y+10xyz+x^2z \overset{?}{\geq} 3yz^2+6xz^2+3z^3$$

$$\Leftrightarrow 6xz(y-z)+3yz(x-z)+z(xy-z^2)\overset{?}{\geq}0 \to \text{true} \because x\geq y\geq z$$

$$\therefore \boxed{\frac{\sqrt{3}}{2}b \leq g_a} \text{ and finally}, g_b^2 \overset{?}{\leq} p_b^2 \overset{\substack{\text{via Bogdan Fustei}\\ \text{and}\\ \text{Mohamed Amine Ajiba}}}{\Leftrightarrow}$$

$$s(s-b)-\frac{(s-b)(c-a)^2}{b} \overset{?}{\leq} s(s-b)+\frac{s(3s+b)(c-a)^2}{(2s+b)^2} \to \text{evidently true}$$

$$\because s>b \therefore \boxed{g_b \leq p_b} \text{ and so}, \frac{\sqrt{3}}{2}a \leq g_b \leq p_b \leq \frac{\sqrt{3}}{2}c \text{ ; and } p_c \leq \frac{\sqrt{3}}{2}b \leq g_a$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$