ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\min\{w_a, w_b, w_c\} \le 3r \le \sqrt{s^2 - 9Rr} \le \max\{w_a, w_b, w_c\}$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$

$c = x + y$ and $s = x + y + z$ and furthermore, for proving

$\max\{w_a, w_b, w_c\} \overset{?}{\ge} \sqrt{s^2 - 9Rr}$, we assume WLOG $w_a = \max\{w_a, w_b, w_c\}$

$\therefore w_a \ge w_b, w_c \Rightarrow a \le b, c \Rightarrow y + z \le z + x, x + y \Rightarrow \boxed{x \ge y, z}$

$$\text{Now, } w_a^2 \overset{?}{\ge} s^2 - 9Rr$$

$$\Leftrightarrow \frac{4(z + x)(x + y)}{(2x + y + z)^2} . x\left(\sum_{cyc} x\right) \overset{?}{\ge} \left(\sum_{cyc} x\right)^2 - \frac{9(y + z)(z + x)(x + y)}{4(x + y + z)}$$

$$\Leftrightarrow 20x^4y + 20x^4z + 20x^3y^2 + 56x^3yz + 20x^3z^2 + 23x^2y^2z + 23x^2yz^2 \boxed{\overset{?}{\ge}}$$
$$15x^2y^3 + 15x^2z^3 + 19xy^4 + 19xz^4 + 10xy^2z^2 + 24xy^3z + 24xyz^3 +$$
$$4y^5 + 4z^5 + 11y^4z + 11yz^4 + 13y^3z^2 + 13y^2z^3$$
$$\Leftrightarrow 15x^2y(x^2 - y^2) + 15x^2z(x^2 - z^2) + 4y(x^4 - y^4) + 4z(x^4 - z^4) +$$
$$19xy^2(x^2 - y^2) + 19xz^2(x^2 - z^2) + 23xy^2z(x - y) + 23xyz^2(x - z) +$$
$$(xy^2 + xz^2)(x^2 - yz) + 8xyz(x^2 - yz) + 13yz(x^3 - y^2z) + 13yz(x^3 - yz^2) +$$
$$11yz(x^3 - y^3) + 11yz(x^3 - z^3) + xy(x^3 - yz^2) + xz(x^3 - y^2z) \overset{?}{\ge} 0$$

$\rightarrow$ true $\because x \ge y, z \therefore \boxed{\boxed{\max\{w_a, w_b, w_c\} \ge \sqrt{s^2 - 9Rr}}}$ and now, for proving

$\min\{w_a, w_b, w_c\} \overset{?}{\le} 3r$, we assume WLOG $w_a = \min\{w_a, w_b, w_c\} \therefore w_a \le w_b, w_c$

$\Rightarrow a \ge b, c \Rightarrow y + z \ge z + x, x + y \Rightarrow \boxed{y, z \ge x}$ and hence, we can assign

$y = x + m$ and $z = x + n$ $(m, n \ge 0)$ and then : $\min\{w_a, w_b, w_c\} \overset{?}{\le} 3r \Leftrightarrow$

$$\frac{4(z + x)(x + y)}{(2x + y + z)^2} . x\left(\sum_{cyc} x\right) \overset{?}{\le} \frac{9xyz}{x + y + z}$$

$$\Leftrightarrow 9(x+\mathrm{m})(x+\mathrm{n})(4x+\mathrm{m}+\mathrm{n})^2 \overset{?}{\geq} 4(3x+\mathrm{m}+\mathrm{n})^2(2x+\mathrm{n})(2x+\mathrm{m})$$

$$\Leftrightarrow 5\mathrm{mn}(\mathrm{m}^2+\mathrm{n}^2)+x(\mathrm{m}^3+\mathrm{n}^3)+10\mathrm{m}^2\mathrm{n}^2+51\mathrm{mn}x(\mathrm{m}+\mathrm{n})+17x^2(\mathrm{m}^2+\mathrm{n}^2)+$$

$$142\mathrm{mn}x^2+48x^3(\mathrm{m}+\mathrm{n}) \boxed{\overset{?}{\geq}} 0 \to \text{true} \because \mathrm{m},\mathrm{n} \geq 0 \therefore \boxed{\boxed{\min\{\mathrm{w}_a,\mathrm{w}_\mathrm{b},\mathrm{w}_\mathrm{c}\} \leq 3\mathrm{r}}} \text{ and}$$

$$\text{finally}, \mathrm{s}^2-9\mathrm{Rr} \overset{\text{Gerretsen}}{\geq} 7\mathrm{Rr}-5\mathrm{r}^2 \overset{\text{Euler}}{\geq} 14\mathrm{r}^2-5\mathrm{r}^2 = 9\mathrm{r}^2 \Rightarrow \boxed{\boxed{\sqrt{\mathrm{s}^2-9\mathrm{Rr}} \geq 3\mathrm{r}}}$$

$$\text{and so}, \min\{\mathrm{w}_a,\mathrm{w}_\mathrm{b},\mathrm{w}_\mathrm{c}\} \leq 3\mathrm{r} \leq \sqrt{\mathrm{s}^2-9\mathrm{Rr}} \leq \max\{\mathrm{w}_a,\mathrm{w}_\mathrm{b},\mathrm{w}_\mathrm{c}\}\ \forall\ \Delta\ \mathrm{ABC},$$

$$'' ='' \text{ iff } \Delta\ \mathrm{ABC} \text{ is equilateral (QED)}$$