ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\min\{m_a, m_b, m_c\} \le \frac{\sqrt{s^2 - 4Rr + 17r^2}}{2} \le \sqrt{s^2 - 7Rr - 4r^2} \le \max\{m_a, m_b, m_c\}$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$ $c = x + y$ and $s = x + y + z$ and furthermore, for proving $\max\{m_a, m_b, m_c\} \overset{?}{\ge} \sqrt{s^2 - 7Rr - 4r^2}$, we assume WLOG $m_a = \max\{m_a, m_b, m_c\}$ $\therefore m_a \ge m_b, m_c \Rightarrow a \le b, c \Rightarrow y + z \le z + x, x + y \Rightarrow \boxed{x \ge y, z}$

$$\text{Now, } m_a^2 \overset{?}{\ge} s^2 - 7Rr - 4r^2 \Leftrightarrow$$

$$\frac{4x(x+y+z)+(y-z)^2}{4} \overset{?}{\ge} \left(\sum_{\text{cyc}} x\right)^2 - \frac{7(y+z)(z+x)(x+y)}{4(x+y+z)} - \frac{4xyz}{x+y+z}$$

$$\Leftrightarrow x^2y + x^2z + 4xyz \overset{?}{\ge} y^3 + z^3 + 2y^2z + 2yz^2$$

$$\Leftrightarrow y(x^2 - y^2) + z(x^2 - z^2) + 2yz\big((x-y)+(x-z)\big) \overset{?}{\ge} 0 \to \text{true} \because x \ge y, z$$

$\therefore \boxed{\boxed{\max\{m_a, m_b, m_c\} \ge \sqrt{s^2 - 7Rr - 4r^2}}}$ and now, for proving $\min\{m_a, m_b, m_c\} \overset{?}{\le} \frac{\sqrt{s^2 - 4Rr + 17r^2}}{2}$, we assume WLOG $m_a = \min\{m_a, m_b, m_c\} \therefore m_a \le m_b, m_c$ $\Rightarrow a \ge b, c \Rightarrow y + z \ge z + x, x + y \Rightarrow \boxed{y, z \ge x}$ and then, $m_a^2 \overset{?}{\le} \frac{s^2 - 4Rr + 17r^2}{4}$

$$\Leftrightarrow \frac{4x(x+y+z)+(y-z)^2}{4} \overset{?}{\le} \frac{(x+y+z)^3 - (y+z)(z+x)(x+y) + 17xyz}{4(x+y+z)}$$

$$\Leftrightarrow y^2z + yz^2 + 5xyz \overset{?}{\ge} x^3 + 2x^2y + 2x^2z + xy^2 + xz^2$$

$$\Leftrightarrow y^2(z-x) + z^2(y-x) + x(yz - x^2) + 2x\big(y(z-x) + z(y-x)\big) \overset{?}{\ge} 0 \to \text{true}$$

$\because y, z \ge x \therefore \boxed{\boxed{\min\{m_a, m_b, m_c\} \le \frac{\sqrt{s^2 - 4Rr + 17r^2}}{2}}}$ and finally,

$$\frac{\sqrt{s^2 - 4Rr + 17r^2}}{2} \overset{?}{\le} \sqrt{s^2 - 7Rr - 4r^2} \Leftrightarrow s^2 \overset{?}{\ge} 8Rr + 11r^2$$

$\Leftrightarrow (s^2 - 16Rr + 5r^2) + 8r(R - 2r) \overset{?}{\ge} 0 \to \text{true} \because s^2 \overset{\text{Gerretsen}}{\ge} 16Rr - 5r^2$ and $R \overset{\text{Euler}}{\ge} 2r \therefore \boxed{\boxed{\frac{\sqrt{s^2 - 4Rr + 17r^2}}{2} \le \sqrt{s^2 - 7Rr - 4r^2}}}$ and so,

$$\min\{m_a, m_b, m_c\} \le \frac{\sqrt{s^2 - 4Rr + 17r^2}}{2} \le \sqrt{s^2 - 7Rr - 4r^2} \le max\{m_a, m_b, m_c\}$$

$\forall\ \Delta\ ABC, '' ='' $ iff Δ ABC is equilateral (QED)