ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$m_a g_a^2 \geq h_a r_b r_c$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2 g_a^4 \overset{?}{\geq} h_a^2 . s^2(s-a)^2 \overset{\text{Bogdan Fustei}}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)-\frac{(s-a)(b-c)^2}{a}\right)^2 \overset{?}{\geq}$$

$$\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)s^2(s-a)^2$$

$$\Leftrightarrow \left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s^2(s-a)^2+\frac{(s-a)^2(b-c)^4}{a^2}-\frac{2s(s-a)^2(b-c)^2}{a}\right) \overset{?}{\geq}$$

$$s^3(s-a)^3-\frac{s^3(s-a)^3}{a^2}.(b-c)^2$$

$$\Leftrightarrow \frac{s(s-a)(b-c)^2}{a^2}-\frac{2s^2(s-a)}{a}+\frac{s^2}{4}+\frac{(b-c)^4}{4a^2}-\frac{s(b-c)^2}{2a}+\frac{s^3(s-a)}{a^2} \overset{?}{\geq} 0$$

$$(\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{(b-c)^4}{4a^2}+\frac{2s^2-3sa}{2a^2}.(b-c)^2+\frac{s^2}{4a^2}.(4s(s-a)-8a(s-a)+a^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow (b-c)^4+2(b-c)^2.s(2s-3a)+s^2(2s-3a)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \left((b-c)^2+s(2s-3a)\right)^2 \overset{?}{\geq} 0 \to \text{true} \therefore m_a g_a^2 \geq h_a r_b r_c \ \forall \Delta \text{ ABC},$$