ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt[3]{g_a w_a n_a} \le \sqrt{p_a m_a}$$

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$$(p_a m_a)^3 \ge \left(s(s-a)+\frac{1}{3}(b-c)^2\right)^3$$

(Reference : Inequality in Triangle by Dang Ngoc Minh – 324; published at www.ssmrmh.ro)

$$\therefore p_a^3 m_a^3 \ge s^3(s-a)^3+\frac{(b-c)^6}{27}+s^2(s-a)^2(b-c)^2+\frac{s(s-a)(b-c)^4}{3} \to (1)$$

$$\text{Again, } n_a^2 g_a^2 w_a^2 \overset{\text{Bogdan Fustei}}{=}$$

$$\left(s(s-a)+\frac{s}{a}(b-c)^2\right)\left(s(s-a)-\frac{s-a}{a}.(b-c)^2\right)\left(s(s-a)-\frac{s(s-a)}{(2s-a)^2}.(b-c)^2\right)$$

$$=\left(s^2(s-a)^2+s(s-a)(b-c)^2-\frac{s(s-a)}{a^2}.(b-c)^4\right)\left(s(s-a)-\frac{s(s-a)}{(2s-a)^2}.(b-c)^2\right)$$

$$=s^3(s-a)^3+s^2(s-a)^2\left(\frac{-s(s-a)}{(2s-a)^2}.(b-c)^2+(b-c)^2-\frac{(b-c)^4}{(2s-a)^2}-\frac{(b-c)^4}{a^2}+\frac{(b-c)^6}{a^2(2s-a)^2}\right)\overset{?}{\le}$$

$$s^3(s-a)^3+\frac{(b-c)^6}{27}+s^2(s-a)^2(b-c)^2+\frac{s(s-a)(b-c)^4}{3} \Leftrightarrow$$

$$(b-c)^6\left(\frac{s^2(s-a)^2}{a^2(2s-a)^2}-\frac{1}{27}\right)-(b-c)^4\left(\frac{s^2(s-a)^2}{(2s-a)^2}+\frac{s^2(s-a)^2}{a^2}+\frac{s(s-a)}{3}\right)-\left(\frac{s^3(s-a)^3}{(2s-a)^2}-s^2(s-a)^2+s^2(s-a)^2\right)(b-c)^2\overset{?}{\le}0$$

$$\Leftrightarrow \overbrace{(27s^4-54s^3a+23s^2a^2+4sa^3-a^4)}^{\sigma_1}(b-c)^4-$$

$$\overbrace{9s(s-a)(12s^4-24s^3a+22s^2a^2-10sa^3+a^4)}^{\sigma_2}(b-c)^2-\overbrace{27a^2s^3(s-a)^3}^{\sigma_3}\ \boxed{\overset{?}{\le}_{(*)}}\ 0$$

$(\because (b-c)^2 \ge 0)$ and it's trivially true if : $\sigma_1 \le 0$ $\left(\because 12s^4-24s^3a+22s^2a^2-10sa^3+a^4 = (s-a)(12s^2(s-a)+10sa^2)+a^4>0 \text{ as } (s-a)>0\right)$ and when : $\sigma_1 > 0$, then, since $\delta = \sigma_2^2+4\sigma_1\sigma_3 =$

$$(s-a)^2(2s-a)^2\left((s-a)^2(108s^3(s-a)+180s^2a^2+44a^4)+44a^5(s-a)+3a^6\right)>0$$

$\therefore$ in order to prove $(*)$, it suffices to prove :

$$2\sigma_1.(b-c)^2 \boxed{\underset{(i)}{\overset{?}{\leq}}} \sigma_2+\sqrt{\delta} \text{ AND } 2\sigma_1.(b-c)^2 \boxed{\underset{(ii)}{\overset{?}{\geq}}} \sigma_2-\sqrt{\delta}$$

$$\text{Now, } 2\sigma_1.(b-c)^2-\sigma_2 < 2\sigma_1.a^2-\sigma_2 \leq \left|2\sigma_1.a^2-\sigma_2\right| \overset{?}{\leq} \sqrt{\delta}$$

$$\Leftrightarrow \delta \overset{?}{\geq} (2\sigma_1.a^2-\sigma_2)^2$$

$$\Leftrightarrow 3645t^{10}-18225t^9+37611t^8-41094t^7+25155t^6-8181t^5+$$

$$965t^4+212t^3-105t^2+17t-1 \overset{?}{\geq} 0 \left(t=\frac{s}{a}\right)$$

$$\Leftrightarrow \begin{pmatrix}27t^4-54t^3+\\23t^2+4t-1\end{pmatrix}\left((t-1)\left((t-1)\begin{pmatrix}135t^3(t-1)+\\63t^2+13\end{pmatrix}+13\right)+1\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t>1 \text{ and } \sigma_1>0 \Rightarrow 2\sigma_1.(b-c)^2-\sigma_2<\sqrt{\delta} \Rightarrow \text{(i) is true (strict inequality)}$$

$$\text{and finally, } 2\sigma_1.(b-c)^2+\sqrt{\delta} \geq \sqrt{\delta} = \sqrt{\sigma_2^2+4\sigma_1\sigma_3} > \sqrt{\sigma_2^2} = \sigma_2$$

$$(\because \sigma_2>0 \text{ as } 12s^4-24s^3a+22s^2a^2-10sa^3+a^4>0; \text{ demonstrated earlier})$$

$$\Rightarrow \text{(ii) is true (strict inequality)} \therefore \text{both (i) and (ii) are true} \Rightarrow (*) \text{ is true}$$

$$\therefore n_a^2g_a^2w_a^2 \leq s^3(s-a)^3+\frac{(b-c)^6}{27}+s^2(s-a)^2(b-c)^2+\frac{s(s-a)(b-c)^4}{3} \overset{\text{via } ①}{\leq}$$

$$p_a^3m_a^3 \Rightarrow \sqrt[3]{g_aw_an_a} \leq \sqrt{p_am_a}\ \forall \Delta ABC, ''='' \text{ iff } b=c \text{ (QED)}$$