ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{w_a m_a} \le \sqrt{\frac{R}{r} - 1} \cdot h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$ $c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations :

$$y^2 + z^2 = x^2(m^2 - 2n), y^3 + z^3 = x^3(m^3 - 3nn),$$
$$y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2), y^5 + z^5 = x^5\left(m((m^2 - 2n)^2 + n^2 - nm^2)\right),$$
$$y^6 + z^6 = x^6((m^3 - 3nn)^2 - 2n^3), \text{ and now, } w_a m_a \le w_a \frac{\sqrt{s^2 - 11r^2}}{4r} . h_a$$
$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Dang Ngoc Minh} - 77;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right) \overset{?}{\le} \left(\frac{R}{r} - 1\right) h_a^2$$
$$\Leftrightarrow \frac{1}{\cos^2 \frac{B-C}{2}} \overset{?}{\le} \frac{16(s-a)(s-b)(s-c)\left((y+z)(z+x)(x+y) - 4xyz\right)^2}{\left(s^3 - 11(s-a)(s-b)(s-c)\right). 16x^2y^2z^2}$$
$$\Leftrightarrow \frac{(y+z)^2(z+x)(x+y)}{(2x+y+z)^2 yz} \overset{?}{\le} \frac{\left((y+z)(z+x)(x+y) - 4xyz\right)^2}{\left((x+y+z)^3 - 11xyz\right). xyz}$$
$$\Leftrightarrow 3x^6(y+z)^2 + 8x^5(y^3+z^3) - 8x^5.yz(y+z) + 7x^4(y^4+z^4) + 14x^4.y^2z^2 - 18x^4.yz(y^2+z^2) + 2x^3(y^5+z^5) - 6x^3.yz(y^3+z^3) + 4x^3.y^2z^2(y+z) - x^2.yz(y^4+z^4) - 5x^2.y^2z^2(y^2+z^2) - 8x^2.y^3z^3 + x.yz(y^5+z^5) + x.y^2z^2(y^3+z^3) - 2x.y^3z^3(y+z) + y^2z^2(y+z)^4 \boxed{\overset{?}{\underset{(1)}{\ge}}} 0 \text{ and via}$$

set of relations "S", to prove (1), it suffices to prove, following simplification :

$$\overbrace{(m^4 - 4m^3 - m^2 + 32m + 64)}^{\Omega_1} n^2 + \overbrace{(m^5 - m^4 - 16m^3 - 46m^2 - 32m)}^{\Omega_2} n + \overbrace{(2m^5 + 7m^4 + 8m^3 + 3m^2)}^{\Omega_3} \boxed{\overset{?}{\underset{(2)}{\ge}}} 0$$

$$\text{Now, } \Omega_1 = \frac{(m^2 - 4m)^2 + (m^2 - 9)^2 + 64m + 47}{2} \overset{m>0}{>} 0 \text{ and furthermore,}$$

we observe that, if : $\Omega_2 \ge 0$, then (2) is trivially true and so, we now focus on : $\Omega_2 < 0$ and then : $\Omega_2 = (m^4 + 5m^3 + 14m^2 + 38m + 196)(m - 6) + 1176 < 0$ $\Rightarrow (m^4 + 5m^3 + 14m^2 + 38m + 196)(m - 6) < -1176 < 0 \Rightarrow \boxed{m < 6} \to (*)$; also LHS of (2) is a quadratic polynomial in "n" with discriminant, $\delta = \Omega_2^2 - 4\Omega_1.\Omega_3 = m^2(m+1)^2(m^6 - 12m^5 - 4m^4 + 48m^3 + 80m^2 + 256)$ and so, when : $m^6 - 12m^5 - 4m^4 + 48m^3 + 80m^2 + 256 \le 0$, then : $\delta \le 0$

$$\mathbf{and}\ ② \text{ is trivially true and so, we now focus on : } P =$$
$$m^6 - 12m^5 - 4m^4 + 48m^3 + 80m^2 + 256 > 0\ (\delta > 0)$$
$$\Rightarrow \left(m^5 - 9m^4 - 31m^3 - 45m^2 - 55m - 165\right)(m-3) > 239 > 0 \rightarrow (\bullet)$$
$$m^5 - 9m^4 - 31m^3 - 45m^2 - 55m - 165 =$$
$$m^4(m-6-3) - 31m^3 - 45m^2 - 55m - 165 \overset{\text{via } (*)}{<} 0 \overset{\text{via } (\bullet)}{\Rightarrow} \boxed{m < 3} \rightarrow (**)$$
$$\text{Now, in order to prove } ②\text{, it suffices to prove : } 2\Omega_1 . n \overset{?}{\leq} -\Omega_2 - \sqrt{\delta} \text{ and } \because$$
$$n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4} \therefore \text{ it suffices to prove : } m^2\Omega_1 \overset{?}{\leq} -2\Omega_2 - 2\sqrt{\delta}$$
$$\Leftrightarrow 2\sqrt{\delta} \boxed{\overset{?}{\underset{③}{\leq}}} m\left(-m^5 + 2m^4 + 3m^3 + 28m + 64\right) \text{ and now,}$$
$$-m^5 + 2m^4 + 3m^3 + 28m + 64 =$$
$$\frac{m^4(8-m)}{4} + \frac{m^3(12-m^2)}{4} + \frac{m(112-m^4)}{4} + \frac{256-m^5}{4} \overset{m<3, \text{via } (**)}{>} 0$$
$$\therefore \text{ in order to prove } ③\text{, it suffices to prove :}$$
$$m^2\left(-m^5 + 2m^4 + 3m^3 + 28m + 64\right)^2 \overset{?}{\geq} 4m^2(m+1)^2\begin{pmatrix} m^6 - 12m^5 - 4m^4 + \\ 48m^3 + 80m^2 + 256 \end{pmatrix} \Leftrightarrow$$
$$\boxed{m^2(m+2)^2(m-2)^2(2m^3(10-m^2) + 2m^2(30-m^3) + m^6 + 2m^4 + m^2 + 96m + 192) \overset{?}{\geq} 0}$$
$$\rightarrow \text{true} \because m < 3, \text{via } (**) \Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true } \therefore \sqrt{w_a m_a} \leq \sqrt{\frac{R}{r} - 1}.\, h_a$$
$$\forall \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$