ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{g_a m_a} \le \sqrt{\frac{3R-r}{5r}} \cdot h_a$$

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Let $x = s-a, y = s-b, z = s-c$ and then : $a = y+z, b = z+x,$ $c = x+y$ and $s = x+y+z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations : $y^2+z^2 = x^2(m^2-2n),$
$y^3+z^3 = x^3(m^3-3nn), y^4+z^4 = x^4((m^2-2n)^2-2n^2),$
$y^5+z^5 = x^5\left(m((m^2-2n)^2+n^2-nm^2)\right), y^6+z^6 = x^6((m^3-3nn)^2-2n^3),$

$$\text{and now, } \sqrt{g_a m_a} \overset{?}{\le} \sqrt{\frac{3R-r}{5r}}.h_a \overset{\text{via Bogdan Fustei}}{\Leftrightarrow}$$

$$\left((s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}\right)\left(s(s-a)+\frac{(b-c)^2}{4}\right) \overset{?}{\le}$$

$$\left(\frac{3}{5}.\frac{(y+z)(z+x)(x+y)}{4xyz}-\frac{1}{5}\right)^2.\frac{16(x+y+z)^2x^2y^2z^2}{(y+z)^4}$$

$$\Leftrightarrow (x^2(y+z)+4xyz)\left(\frac{4x(x+y+z)+(y-z)^2}{4}\right) \overset{?}{\le}$$

$$\frac{(3(y+z)(z+x)(x+y)-4xyz)^2}{400}.\frac{16(x+y+z)^2}{(y+z)^3}$$

$$\Leftrightarrow 36x^6(y+z)^2+144x^5(y^3+z^3)+336x^5.yz(y+z)+116x^4(y^4+z^4)+$$
$$248x^4.yz(y^2+z^2)+328x^4.y^2z^2+44x^3(y^5+z^5)-252x^3.yz(y^3+z^3)-$$
$$944x^3.y^2z^2(y+z)+11x^2(y^6+z^6)-114x^2.yz(y^4+z^4)-647x^2.y^2z^2(y^2+z^2)$$
$$-1044x^2.y^3z^3-8x.yz(y^5+z^5)+236x.y^2z^2(y^3+z^3)+848x.y^3z^3(y+z)+$$
$$36y^2z^2(y+z)^4 \boxed{\underset{(1)}{\overset{?}{\ge}}} 0$$

and via set of relations "S", in order to prove (1),

suffices to prove, via simplification : $\overbrace{(36m^4+376m^3-92m^2+32m+64)}^{\sigma_1}n^2 -$

$$\overbrace{(28m^5+180m^4+472m^3+216m^2+96m)}^{\sigma_2}n +$$

$$\overbrace{(11m^6+44m^5+116m^4+144m^3+36m^2)}^{\sigma_3} + \boxed{\underset{(2)}{\overset{?}{\ge}}} 0$$

Now, discriminant of : $376m^2-92m+32 = (92)^2-(128)(376) < 0$
$\Rightarrow 376m^3-92m^2+32m > 0 \Rightarrow \sigma_1 > 0$ and $\because$ LHS of (2) is a quadratic polynomial in "n" with discriminant, $\boxed{\delta} = \sigma_2^2 - 4\sigma_1\sigma_3 =$
$-800m^4(m+1)^2(m^4+14m^3-4m^2-8m-128), \therefore$ (2) is trivially true if :

$$m^4+14m^3-4m^2-8m-128\ge 0\ (making\ \delta\le 0)\ and\ \text{when}:$$

$$\boxed{m^4+14m^3-4m^2-8m-128\overset{(*)}{<}0}\ (\delta>0),\text{then, in order to prove ②,}$$

it suffices to prove : $2\sigma_1.n\overset{?}{\le}\sigma_2-\sqrt{\delta}$ & since $n\overset{AM-GM}{\le}\frac{m^2}{4}$ ∴ it suffices to prove :

$$\sigma_1.\frac{m^2}{2}\overset{?}{\le}\sigma_2-\sqrt{\delta}\Leftrightarrow -2m(m+1)(9m^4+71m^3-184m^2-44m-48)\boxed{\overset{?}{\underset{③}{\ge}}}$$

$$\sqrt{-800m^4(m+1)^2(m^4+14m^3-4m^2-8m-128)}$$

Now, $m^4+14m^3-4m^2-8m-128<0$ (we are considering) $\Rightarrow$

$$\frac{1}{4}\left(m^3+\frac{65m^2}{4}+\frac{521m}{16}+\frac{4177}{64}\right)(4m-9)+\frac{4825}{256}<0$$

$$\Rightarrow\frac{1}{4}\left(m^3+\frac{65m^2}{4}+\frac{521m}{16}+\frac{4177}{64}\right)(4m-9)<-\frac{4825}{256}<0\Rightarrow 4m-9<0\to(\bullet)$$

We have : $9m^4+71m^3-184m^2-44m-48=$

$$\left(\frac{4m-9}{4}\right)\left(9m^3+\frac{365m^2}{4}+\frac{341m}{16}+\frac{253}{64}\right)-\frac{10011}{64}\overset{via\,(*)}{<}-\frac{10011}{64}<0$$

$$\Rightarrow\boxed{-2m(m+1)(9m^4+71m^3-184m^2-44m-48)\overset{(**)}{>}0}$$

∴ $(*)$ and $(**)$ $\Rightarrow$ in order to prove ③, it suffices to prove :

$$4m^2(m+1)^2(9m^4+71m^3-184m^2-44m-48)^2\overset{?}{\ge}$$
$$-800m^4(m+1)^2(m^4+14m^3-4m^2-8m-128)$$

$$\Leftrightarrow\boxed{12m^2(m+1)^2(m-2)^2(9m+4)(3m^5+58m^4+271m^3-64m^2+28m+48)\overset{?}{\ge}0}$$

$\to$ true ∵ discriminant of : $271m^2-64m+28=(64)^2-(112)(271)<0$

$$\Rightarrow 271m^3-64m^2+28m>0\Rightarrow 3m^5+58m^4+271m^3-64m^2+28m+48>0$$

$$\Rightarrow ③\Rightarrow ②\Rightarrow ① \text{ is true } \therefore \sqrt{g_a m_a}\le\sqrt{\frac{3R-r}{5r}}.h_a\ \forall\,\Delta\,ABC,$$

$''=''$ iff $b=c$ and $b+c=2a\Rightarrow ''=''$ iff $\Delta\,ABC$ is equilateral (QED)