ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$max\{r_a, r_b, r_c\} \geq \sqrt{4R^2 - 7r^2} \geq \frac{r(s^2 + 4Rr + r^2)}{s^2 - 8Rr + r^2} \geq \min\{r_a, r_b, r_c\}$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$

$c = x + y$ & $s = x + y + z$; furthermore, for proving $max\{r_a, r_b, r_c\} \overset{?}{\geq} \sqrt{4R^2 - 7r^2}$,

we assume WLOG $r_a = max\{r_a, r_b, r_c\} \therefore r_a \geq r_b, r_c \Rightarrow a \geq b, c$

$\Rightarrow y + z \geq z + x, x + y \Rightarrow \boxed{y, z \geq x}$ and hence, we can assign :

$y = x + m$ and $z = x + n$ $(m, n \geq 0)$ and then : $max\{r_a, r_b, r_c\} \overset{?}{\geq} \sqrt{4R^2 - 7r^2}$

$$\Leftrightarrow \frac{s^2}{(s-a)^2} \overset{?}{\geq} \frac{4R^2}{r^2} - 7 \Leftrightarrow \frac{(x+y+z)^2}{x^2} \overset{?}{\geq} \frac{(y+z)^2(z+x)^2(x+y)^2}{4x^2y^2z^2} - 7$$

$$\Leftrightarrow 4(x+m)^2(x+n)^2(3x+m+n)^2 + 28x^2(x+m)^2(x+n)^2 -$$

$$(3x+m+n)^2(2x+m)^2 2(x+n)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 3m^4n^2 + 4m^4nx + 6m^3n^3 + 32m^3n^2x + 32m^3nx^2 + 3m^2n^4 + 32m^2n^3x +$$

$$124m^2n^2x + 128m^2nx^3 + 20m^2x^4 + 4mn^4x + 32mn^3x^2 + 128mn^2x^3 +$$

$$136mnx^4 + 24mx^5 + 20n^2x^4 + 24nx^5 \overset{?}{\geq} 0 \rightarrow \text{true} \because m, n \geq 0$$

$\therefore \boxed{max\{r_a, r_b, r_c\} \geq \sqrt{4R^2 - 7r^2}}$ and now, for proving $\frac{r(s^2 + 4Rr + r^2)}{s^2 - 8Rr + r^2} \overset{?}{\geq} \min\{r_a, r_b, r_c\}$,

we assume WLOG $r_a = \min\{r_a, r_b, r_c\} \therefore r_a \leq r_b, r_c \Rightarrow a \leq b, c$

$\Rightarrow y + z \leq z + x, x + y \Rightarrow \boxed{y, z \leq x}$ and then : $\frac{r(s^2 + 4Rr + r^2)}{s^2 - 8Rr + r^2} \overset{?}{\geq} \min\{r_a, r_b, r_c\}$

$$\Leftrightarrow \frac{r\left(\left(\sum_{cyc} x\right)^2 + (y+z)(z+x)(x+y) + xyz\right)}{\left(\sum_{cyc} x\right)^2 - 2(y+z)(z+x)(x+y) + xyz} \overset{?}{\geq} \frac{r(x+y+z)}{x}$$

$$\Leftrightarrow 2x^3(y+z) + 2x^2(y^2 + z^2 + 2yz) - x(y^3 + z^3 + y^2z + yz^2) -$$

$$y^4 - z^4 - 2y^2z^2 - 2y^3z - 2yz^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow 2y(x^3 - z^3) + 2z(x^3 - y^3) + y^2(x^2 - y^2) + z^2(x^2 - z^2) +$$

$$2yz(x^2 - yz) + x^2(y+z)^2 - x(y+z)(y^2 + z^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow 2y(x^3 - z^3) + 2z(x^3 - y^3) + y^2(x^2 - y^2) + z^2(x^2 - z^2) + 2yz(x^2 - yz) +$$

$$x(y+z)\big(y(x-y) + z(x-z)\big) \overset{?}{\geq} 0 \rightarrow \text{true} \because x \geq y, z$$

$\therefore \boxed{\frac{r(s^2 + 4Rr + r^2)}{s^2 - 8Rr + r^2} \geq \min\{r_a, r_b, r_c\}}$ and finally, $\frac{r(s^2 + 4Rr + r^2)}{s^2 - 8Rr + r^2} \overset{\text{Gerretsen}}{\leq}$

$$\frac{r(4R^2 + 8Rr + 4r^2)}{8Rr - 4r^2} \overset{?}{\leq} \sqrt{4R^2 - 7r^2} \Leftrightarrow 15t^4 - 20t^3 - 30t^2 + 24t - 8 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(15t^3+10t^2-10t+4) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \boxed{\sqrt{4R^2-7r^2} \geq \frac{r(s^2+4Rr+r^2)}{s^2-8Rr+r^2}} \text{ and so,}$$

$$\max\{r_a, r_b, r_c\} \geq \sqrt{4R^2-7r^2} \geq \frac{r(s^2+4Rr+r^2)}{s^2-8Rr+r^2} \geq \min\{r_a, r_b, r_c\} \ \forall \Delta ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)