ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\max\{a,b,c\} \geq 2s.\frac{R-r}{2R-r} \geq \frac{3abc}{ab+bc+ca} \geq \min\{a,b,c\}$$

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Let $x = s-a, y = s-b, z = s-c$ and then : $a = y+z, b = z+x,$ $c = x+y$ & $s = x+y+z$ and furthermore, for proving $\max\{a,b,c\} \geq 2s.\frac{R-r}{2R-r}$, we assume WLOG $a = \max\{a,b,c\} \therefore a \geq b,c \Rightarrow a \geq b,c \Rightarrow y+z \geq z+x, x+y$ $\Rightarrow \boxed{y,z \geq x}$ and then : $\max\{a,b,c\} \overset{?}{\geq} 2s.\frac{R-r}{2R-r}$

$$\Leftrightarrow y+z \overset{?}{\geq} \frac{(x+y+z)\big((y+z)(z+x)(x+y)-4xyz\big)}{(y+z)(z+x)(x+y)-2xyz}$$

$$\Leftrightarrow (x^2yz-x^3y)+(x^2yz-x^3z)+(xy^2z-x^2y^2)+(xyz^2-x^2z^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow x^2y(z-x)+x^2z(y-x)+xy^2(z-x)+xz^2(y-x) \overset{?}{\geq} 0 \to \text{true} \because y,z \geq x$$

$\therefore \boxed{\max\{a,b,c\} \geq 2s.\frac{R-r}{2R-r}}$ and now, for proving $\frac{3abc}{ab+bc+ca} \overset{?}{\geq} \min\{a,b,c\}$, we assume WLOG $a = \min\{a,b,c\} \therefore a \leq b,c \Rightarrow y+z \leq z+x, x+y$ $\Rightarrow \boxed{y,z \leq x}$ and then : $\frac{3abc}{ab+bc+ca} \overset{?}{\geq} \min\{a,b,c\}$

$$\Leftrightarrow \frac{3(y+z)(z+x)(x+y)}{(y+z)(z+x)+(z+x)(x+y)+(x+y)(y+z)} \overset{?}{\geq} y+z \Leftrightarrow 2x^2 \overset{?}{\geq} y^2+z^2$$

$$\Leftrightarrow (x^2-y^2)+(x^2-z^2) \overset{?}{\geq} 0 \to \text{true} \because x \geq y,z \therefore \boxed{\frac{3abc}{ab+bc+ca} \geq \min\{a,b,c\}}$$

and finally, $\frac{3abc}{ab+bc+ca} \overset{\text{Gerretsen}}{\leq} \frac{12Rrs}{20Rr-4r^2} \overset{?}{\leq} 2s.\frac{R-r}{2R-r}$

$\Leftrightarrow (4R-r)(R-2r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \therefore \boxed{2s.\frac{R-r}{2R-r} \geq \frac{3abc}{ab+bc+ca}}$ and so,

$$\max\{a,b,c\} \geq 2s.\frac{R-r}{2R-r} \geq \frac{3abc}{ab+bc+ca} \geq \min\{a,b,c\} \ \forall \Delta \text{ ABC},$$

" = " iff Δ ABC is equilateral (QED)