ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{h_a}{r_a+r_b}+\frac{h_b}{r_b+r_c}+\frac{h_c}{r_c+r_a}\le\frac{2R^2+r^2}{R(R+r)}$$

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$$\frac{h_a}{r_a+r_b}+\frac{h_b}{r_b+r_c}+\frac{h_c}{r_c+r_a}=$$

$$=\frac{1}{\prod_{cyc}\left(4R\cos^2\frac{A}{2}\right)}\cdot\sum_{cyc}\frac{2rs.\left(4R\cos^2\frac{A}{2}\right)\left(4R\cos^2\frac{B}{2}\right)}{4R\left(\cos^2\frac{A}{2}\right)\left(\tan\frac{A}{2}\right)}=$$

$$=\frac{4R}{64R^3.\frac{s^2}{16R^2}}\cdot\sum_{cyc}\left(\frac{2r_ar_br_c}{r_a}\cdot\frac{sb(s-b)}{abc}\right)=\frac{2s^2}{abcs^2}\sum_{cyc}\left(b(s-a)(s-b)\right)=$$

$$\frac{2}{(y+z)(z+x)(x+y)}\cdot\sum_{cyc}(xy(z+x))\ \left(\begin{matrix}x=s-a,y=s-b,z=s-c\ and\ then:\\ a=y+z,b=z+x,c=x+y\end{matrix}\right)$$

$$\overset{?}{\le}\frac{2R^2+r^2}{R(R+r)}=\frac{\frac{2R^2}{r^2}+1}{\frac{R^2}{r^2}+\frac{R}{r}}=\frac{\frac{(y+z)^2(z+x)^2(x+y)^2}{8x^2y^2z^2}+1}{\frac{(y+z)^2(z+x)^2(x+y)^2}{16x^2y^2z^2}+\frac{(y+z)(z+x)(x+y)}{4xyz}}$$

$$\Leftrightarrow xyz\sum_{cyc}x^3+\sum_{cyc}x^2y^4+xyz\sum_{cyc}xy^2\underset{(*)}{\overset{?}{\ge}}3xyz\sum_{cyc}x^2y$$

Now, $m^2+n^2+p^2\ge mn+np+pm$ and choosing $m\equiv xy^2,n\equiv yz^2,p\equiv zx^2$,

we get : $\sum_{cyc}x^2y^4\ge xyz\sum_{cyc}x^2y$ and since : $\sum_{cyc}x^3+\sum_{cyc}xy^2\overset{AM-GM}{\ge}2\sum_{cyc}x^2y$

$\therefore$ LHS of $(*)\ge 2xyz\sum_{cyc}x^2y+xyz\sum_{cyc}x^2y\Rightarrow(*)$ is true

$$\therefore\frac{h_a}{r_a+r_b}+\frac{h_b}{r_b+r_c}+\frac{h_c}{r_c+r_a}\le\frac{2R^2+r^2}{R(R+r)}\ \forall\,\Delta\,ABC,$$

'' ='' iff Δ ABC is equilateral (QED)