ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\max\{a,b,c\} \ge 2s.\frac{R-r}{2R-r} \ge \frac{3abc}{ab+bc+ca} \ge \min\{a,b,c\}$$

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$$\text{Let } x = s-a, y = s-b, z = s-c \text{ and then : } a = y+z, b = z+x,$$
$$c = x+y \;\&\; s = x+y+z \text{ and furthermore, for proving } \max\{a,b,c\} \ge 2s.\frac{R-r}{2R-r},$$
$$\text{we assume WLOG } a = \max\{a,b,c\} \therefore a \ge b,c \Rightarrow a \ge b,c \Rightarrow y+z \ge z+x, x+y$$
$$\Rightarrow \boxed{y,z \ge x} \text{ and then : } \max\{a,b,c\} \overset{?}{\ge} 2s.\frac{R-r}{2R-r}$$
$$\Leftrightarrow y+z \overset{?}{\ge} \frac{(x+y+z)\big((y+z)(z+x)(x+y)-4xyz\big)}{(y+z)(z+x)(x+y)-2xyz}$$
$$\Leftrightarrow (x^2yz - x^3y) + (x^2yz - x^3z) + (xy^2z - x^2y^2) + (xyz^2 - x^2z^2) \overset{?}{\ge} 0$$
$$\Leftrightarrow x^2y(z-x) + x^2z(y-x) + xy^2(z-x) + xz^2(y-x) \overset{?}{\ge} 0 \to \text{true} \because y,z \ge x$$
$$\therefore \boxed{\max\{a,b,c\} \ge 2s.\frac{R-r}{2R-r}} \text{ and now, for proving } \frac{3abc}{ab+bc+ca} \overset{?}{\ge} \min\{a,b,c\},$$
$$\text{we assume WLOG } a = \min\{a,b,c\} \therefore a \le b,c \Rightarrow y+z \le z+x, x+y$$
$$\Rightarrow \boxed{y,z \le x} \text{ and then : } \frac{3abc}{ab+bc+ca} \overset{?}{\ge} \min\{a,b,c\}$$
$$\Leftrightarrow \frac{3(y+z)(z+x)(x+y)}{(y+z)(z+x)+(z+x)(x+y)+(x+y)(y+z)} \overset{?}{\ge} y+z \Leftrightarrow 2x^2 \overset{?}{\ge} y^2+z^2$$
$$\Leftrightarrow (x^2-y^2)+(x^2-z^2) \overset{?}{\ge} 0 \to \text{true} \because x \ge y,z \therefore \boxed{\frac{3abc}{ab+bc+ca} \ge \min\{a,b,c\}}$$
$$\text{and finally, } \frac{3abc}{ab+bc+ca} \overset{\text{Gerretsen}}{\le} \frac{12Rrs}{20Rr-4r^2} \overset{?}{\le} 2s.\frac{R-r}{2R-r}$$
$$\Leftrightarrow (4R-r)(R-2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \therefore \boxed{2s.\frac{R-r}{2R-r} \ge \frac{3abc}{ab+bc+ca}} \text{ and so,}$$
$$\max\{a,b,c\} \ge 2s.\frac{R-r}{2R-r} \ge \frac{3abc}{ab+bc+ca} \ge \min\{a,b,c\} \;\forall \Delta \text{ ABC},$$
$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$