

In any ΔABC holds :

$$\frac{w_a h_a}{r_a} + \frac{w_b h_b}{r_b} + \frac{w_c h_c}{r_c} \geq n_a + n_b + n_c$$

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$$\begin{aligned} \sum_{cyc} \frac{w_a^2 h_a^2}{r_a^2} &= \frac{1}{4R^2 r^2} \cdot \sum_{cyc} \left((b^2 c^2 + 16R^2 r^2 - 8Rrbc) \left(bc - \frac{a^2 bc}{(b+c)^2} \right) \right) \rightarrow \textcircled{1} \\ -a^2 b^2 c^2 \sum_{cyc} \frac{bc}{(b+c)^2} &= \frac{-16R^2 r^2 s^2 \cdot \sum_{cyc} \left(bc \left(a^4 + 2a^2 \sum_{cyc} ab + (\sum_{cyc} ab)^2 \right) \right)}{4s^2 (s^2 + 2Rr + r^2)^2} \\ &= \frac{-4R^2 r^2 (s^6 + (36Rr + 3r^2)s^4 + r^2(64R^2 + 16Rr + 3r^2)s^2 + r^3(4R + r)^3)}{(s^2 + 2Rr + r^2)^2} \\ &= X_1 = -a^2 b^2 c^2 \sum_{cyc} \frac{bc}{(b+c)^2} \rightarrow \textcircled{2}; \text{ also, } -16R^2 r^2 \cdot 4Rrs \sum_{cyc} \frac{a}{(b+c)^2} = \\ &\quad - \frac{64R^3 r^3}{(s^2 + 2Rr + r^2)^2} (2s^4 - (3Rr + 6r^2)s^2 + Rr^2(4R + r)) = X_2 = \\ &\quad 16R^2 r^2 \cdot 4Rrs \sum_{cyc} \frac{a}{(b+c)^2} \rightarrow \textcircled{3}; \text{ Also, } 8Rr \cdot 16R^2 r^2 s^2 \cdot \sum_{cyc} \frac{1}{(b+c)^2} = \\ &\quad \frac{32R^3 r^3}{(s^2 + 2Rr + r^2)^2} \cdot \left(2 \sum_{cyc} a^2 b^2 - 16r^2 s^2 + 2 \left(\sum_{cyc} ab \right) \left(\sum_{cyc} a^2 \right) + 3 \left(\sum_{cyc} ab \right)^2 \right) \\ &= \frac{32R^3 r^3}{(s^2 + 2Rr + r^2)^2} \cdot (9s^4 + (8Rr - 6r^2)s^2 + r^2(4R + r)^2) = X_3 = \\ &\quad 8Rr \cdot 16R^2 r^2 s^2 \cdot \sum_{cyc} \frac{1}{(b+c)^2} \rightarrow \textcircled{4} \& \therefore \sum_{cyc} \frac{r_a}{h_a w_a} \geq \sum_{cyc} \frac{r_a}{w_a^2} \therefore 2 \sum_{cyc} \frac{w_b w_c h_b h_c}{r_b r_c} \geq \\ &\quad \frac{2 \cdot 16Rr^2 s^2}{s^2 + 2Rr + r^2} \cdot \frac{2r^2 s^2}{R \cdot s^2 r} \cdot \sum_{cyc} \frac{r \left(s^2 + (s-a)^2 + 2s(s-a) \right)}{4bc(s-a)^2} \rightarrow \textcircled{5} \\ \text{Now } \therefore (s-b)^2 (s-c)^2 &= a^2 s^2 + 2abcs - 2as^3 + b^2 c^2 - 2bcs^2 + s^4 \\ \therefore \frac{r}{4abc} &\left(\frac{s^2}{r^4 s^2} \sum_{cyc} (a(s-b)^2 (s-c)^2) + 2s + \frac{2}{r^2} \sum_{cyc} (a(s-b)(s-c)) \right) \\ &= \frac{r \left(\frac{s^2 \sum_{cyc} a^3 + 2abcs(2s) - 2s^3 \sum_{cyc} a^2 + abc \sum_{cyc} ab - 6abcs^2 + 2s^5}{r^4} + 2s + \frac{2}{r^2} (-2s^3 + s \sum_{cyc} a^2 + 12Rrs) \right)}{16Rrs} \\ \therefore \textcircled{5} \Rightarrow 2 \sum_{cyc} \frac{w_b w_c h_b h_c}{r_b r_c} &\geq \frac{8rs^2(8R^2 + 6Rr - r^2 - s^2)}{R(s^2 + 2Rr + r^2)} = X_4 \rightarrow \textcircled{6} \text{ and } \therefore X_5 = \end{aligned}$$

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$$\sum_{cyc} a^3 b^3 = (s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2) \therefore \textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}, \textcircled{5}, \textcircled{6} \Rightarrow$$

$$\left(\sum_{cyc} \frac{w_a h_a}{r_a} \right)^2 \geq X_1 + X_2 + X_3 + X_4 + \frac{X_5 + 16R^2 r^2 \sum_{cyc} ab - 8Rr \sum_{cyc} a^2 b^2}{4R^2 r^2} \stackrel{?}{\geq}$$

$$\frac{(9R - 3r)s^2 - 3r(4R + r)^2}{R} \Leftrightarrow s^{10} - (16Rr - 5r^2)s^8 -$$

$$r^2(36R^2 + 60Rr - 10r^2)s^6 + r^3(560R^3 + 216R^2r - 52Rr^2 + 10r^3)s^4 +$$

$$r^4(1648R^4 + 1696R^3r + 412R^2r^2 + 12Rr^3 + 5r^4)s^2 +$$

$$r^5(768R^5 + 1152R^4r + 624R^3r^2 + 160R^2r^3 + 20Rr^4 + r^5) \stackrel{?}{\geq} 0 \text{ and } \therefore P =$$

$$T^5 + 4r(16R - 5r)T^4 + 4r^2(375R^2 - 255Rr + 40r^2)T^3 +$$

$$4r^3(3804R^3 - 4371R^2r + 1532Rr^2 - 160r^3)T^2 \stackrel{\text{Gerretsen}}{\geq} 0 \quad (T = s^2 - 16Rr + 5r^2)$$

(via successively implementing Gerretsen) $\therefore$ to prove (*), suffices to prove :

$$\text{LHS of } (*) \stackrel{?}{\geq} P \Leftrightarrow (3591R^4 - 6732R^3r + 4298R^2r^2 - 1028Rr^3 + 80r^4)s^2 \stackrel{?}{\geq} \sum_{(**)}$$

$$2r(28008R^5 - 59028R^4r + 45182R^3r^2 - 16115R^2r^3 + 2690Rr^4 - 168r^5) \&$$

$$\text{LHS}_{(**)} \stackrel{\text{Rouche} + \text{Euler}}{\geq} \left(\frac{3591R^4 - 6732R^3r + 4298R^2r^2}{1028Rr^3 + 80r^4} - \left(\frac{16Rr - 5r^2}{R - r} \right)^2 \right) \stackrel{?}{\geq}$$

$$\text{RHS}_{(**)} \Leftrightarrow 1440t^6 - 5460t^5 + 5761t^4 - 10t^3 - 2876t^2 + 1032t - 96 \stackrel{?}{\geq} 0$$

$$\left(t = \frac{R}{r} \right) \Leftrightarrow (t - 2)(150t^5 + 1290t^4(t - 2) + 601t^3 + 1192t^2 - 492t + 48) \stackrel{?}{\geq} 0$$

$$\rightarrow \text{true } \therefore t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \left(\sum_{cyc} \frac{w_a h_a}{r_a} \right)^2 \geq \frac{(9R - 3r)s^2 - 3r(4R + r)^2}{R} \rightarrow (m)$$

$$\text{Again, } \sum_{cyc} n_a \stackrel{\text{Bogdan Fustei}}{=} \sqrt{s^2 - \frac{rs^2}{R} \cdot \sec^2 \frac{A}{2}} \stackrel{\text{CBS}}{\leq} \sqrt{3 \sum_{cyc} \left(s^2 - \frac{rs^2}{R} \cdot \sec^2 \frac{A}{2} \right)}$$

$$= \sqrt{\frac{(9R - 3r)s^2 - 3r(4R + r)^2}{R}} \rightarrow (n) \therefore (m) \text{ and } (n) \Rightarrow \text{QED, " = " iff } a = b = c$$