ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC holds :

$$\frac{w_a h_a}{r_a}+\frac{w_b h_b}{r_b}+\frac{w_c h_c}{r_c}\geq p_a.\sqrt{\frac{w_a}{g_a}}+p_b.\sqrt{\frac{w_b}{g_b}}+p_c.\sqrt{\frac{w_c}{g_c}}$$

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$$\sum_{cyc}\frac{w_a^2h_a^2}{r_a^2}=\frac{1}{4R^2r^2}.\sum_{cyc}\left((b^2c^2+16R^2r^2-8Rrbc)\left(bc-\frac{a^2bc}{(b+c)^2}\right)\right)\to ①$$

$$-a^2b^2c^2\sum_{cyc}\frac{bc}{(b+c)^2}=\frac{-16R^2r^2s^2.\sum_{cyc}\left(bc\left(a^4+2a^2\sum_{cyc}ab+\left(\sum_{cyc}ab\right)^2\right)\right)}{4s^2(s^2+2Rr+r^2)^2}$$

$$=\frac{-4R^2r^2(s^6+(36Rr+3r^2)s^4+r^2(64R^2+16Rr+3r^2)s^2+r^3(4R+r)^3)}{(s^2+2Rr+r^2)^2}$$

$$=X_1=-a^2b^2c^2\sum_{cyc}\frac{bc}{(b+c)^2}\to ②;\ also,-16R^2r^2.4Rrs\sum_{cyc}\frac{a}{(b+c)^2}=$$

$$-\frac{64R^3r^3}{(s^2+2Rr+r^2)^2}\left(2s^4-(3Rr+6r^2)s^2+Rr^2(4R+r)\right)=X_2=$$

$$16R^2r^2.4Rrs\sum_{cyc}\frac{a}{(b+c)^2}\to ③;\ Also,8Rr.16R^2r^2s^2.\sum_{cyc}\frac{1}{(b+c)^2}=$$

$$\frac{32R^3r^3}{(s^2+2Rr+r^2)^2}.(9s^4+(8Rr-6r^2)s^2+r^2(4R+r)^2)=X_3=$$

$$8Rr.16R^2r^2s^2.\sum_{cyc}\frac{1}{(b+c)^2}\to ④\ \&\ \because\sum_{cyc}\frac{r_a}{h_aw_a}\geq\sum_{cyc}\frac{r_a}{w_a^2}\therefore 2\sum_{cyc}\frac{w_bw_ch_bh_c}{r_br_c}\geq$$

$$\frac{2.16Rr^2s^2}{s^2+2Rr+r^2}.\frac{2r^2s^2}{R.s^2r}.\sum_{cyc}\frac{r\left(s^2+(s-a)^2+2s(s-a)\right)}{4bc(s-a)^2}\to ⑤$$

$$\text{Now}\ \because (s-b)^2(s-c)^2=a^2s^2+2abcs-2as^3+b^2c^2-2bcs^2+s^4$$

$$\therefore\frac{r}{4abc}\left(\frac{s^2}{r^4s^2}\sum_{cyc}(a(s-b)^2(s-c)^2)+2s+\frac{2}{r^2}\sum_{cyc}(a(s-b)(s-c))\right)$$

$$=\frac{r\left(\frac{s^2\sum_{cyc}a^3+2abcs(2s)-2s^3\sum_{cyc}a^2+abc\sum_{cyc}ab-6abcs^2+2s^5}{r^4}+2s+\frac{2}{r^2}\left(-2s^3+s\sum_{cyc}a^2+12Rrs\right)\right)}{16Rrs}$$

$$\therefore ⑤\Rightarrow 2\sum_{cyc}\frac{w_bw_ch_bh_c}{r_br_c}\geq\frac{8rs^2(8R^2+6Rr-r^2-s^2)}{R(s^2+2Rr+r^2)}=X_4\to ⑥\ and\ \because X_5=$$

$$\sum_{cyc} a^3b^3 = (s^2+4Rr+r^2)^3 - 24Rrs^2(s^2+2Rr+r^2) \therefore ①,②,③,④,⑤,⑥ \Rightarrow$$

$$\left(\sum_{cyc}\frac{w_a h_a}{r_a}\right)^2 \geq X_1+X_2+X_3+X_4+\frac{X_5+16R^2r^2\sum_{cyc}ab-8Rr\sum_{cyc}a^2b^2}{4R^2r^2} \overset{?}{\geq}$$

$$\frac{(9R-3r)s^2-3r(4R+r)^2}{R} \Leftrightarrow s^{10}-(16Rr-5r^2)s^8-$$

$$r^2(36R^2+60Rr-10r^2)s^6+r^3(560R^3+216R^2r-52Rr^2+10r^3)s^4+$$

$$r^4(1648R^4+1696R^3r+412R^2r^2+12Rr^3+5r^4)s^2+$$

$$r^5\left(768R^5+1152R^4r+624R^3r^2+160R^2r^3+20Rr^4+r^5\right) \boxed{\overset{?}{\geq} (*)} 0 \text{ and } \because P =$$

$$T^5+4r(16R-5r)T^4+4r^2(375R^2-255Rr+40r^2)T^3+$$

$$4r^3(3804R^3-4371R^2r+1532Rr^2-160r^3)T^2 \overset{\text{Gerretsen}}{\geq} 0 \ (T=s^2-16Rr+5r^2)$$

(*via successively* implementing Gerretsen) $\therefore$ to prove $(*)$, suffices to prove :

$$\text{LHS of } (*) \overset{?}{\geq} P \Leftrightarrow (3591R^4-6732R^3r+4298R^2r^2-1028Rr^3+80r^4)s^2 \boxed{\overset{?}{\geq} (**)}$$

$$2r\left(28008R^5-59028R^4r+45182R^3r^2-16115R^2r^3+2690Rr^4-168r^5\right) \ \&$$

$$\text{LHS}_{(**)} \overset{\text{Rouche + Euler}}{\geq} \left(\begin{matrix}3591R^4-6732R^3r+4298R^2r^2-\\1028Rr^3+80r^4\end{matrix}\right)\left(\begin{matrix}16Rr-5r^2+\\ \dfrac{r^2(R-2r)}{R-r}\end{matrix}\right) \overset{?}{\geq}$$

$$\text{RHS}_{(**)} \Leftrightarrow 1440t^6-5460t^5+5761t^4-10t^3-2876t^2+1032t-96 \overset{?}{\geq} 0$$

$$\left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)\left(150t^5+1290t^4(t-2)+601t^3+1192t^2-492t+48\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow \left(\sum_{cyc}\frac{w_a h_a}{r_a}\right)^2 \geq \frac{(9R-3r)s^2-3r(4R+r)^2}{R} \rightarrow (m)$$

$$\text{Again}, \sum_{cyc} n_a \overset{\text{Bogdan Fustei}}{=} \sqrt{s^2-\frac{rs^2}{R}.\sec^2\frac{A}{2}} \overset{\text{CBS}}{\leq} \sqrt{3\sum_{cyc}\left(s^2-\frac{rs^2}{R}.\sec^2\frac{A}{2}\right)}$$

$$= \sqrt{\frac{(9R-3r)s^2-3r(4R+r)^2}{R}} \rightarrow (n) \therefore (m),(n) \Rightarrow \sum_{cyc}\frac{w_a h_a}{r_a} \geq \sum_{cyc} n_a \geq$$

$$\sum_{cyc} p_a.\sqrt{\frac{w_a}{g_a}} \left(\begin{matrix}\text{Reference : Solution to Inequality in Triangle} - 136\\ \text{by Bogdan Fustei; published at www.ssmrmh.ro}\end{matrix}\right) \forall \Delta ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$