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In any ΔABC the following relationship holds :

$$\sum_{cyc} \sqrt{\frac{1}{2} \left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1 \right)} \leq \sqrt{\frac{\max(r_a, r_b, r_c) + AI + BI + CI}{r}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum_{cyc} \sqrt{\frac{1}{2} \left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1 \right)} &= \sum_{cyc} \sqrt{\frac{1}{2} \left(\frac{n_b n_c}{r_b r_c} + \frac{2r_b h_b + 2r_c h_c}{2r_b r_c} - 1 \right)} \\ &\stackrel{\text{Bogdan Fustei}}{=} \sum_{cyc} \sqrt{\frac{1}{2} \left(\frac{2n_b n_c}{2r_b r_c} + \frac{s^2 - n_b^2 + s^2 - n_c^2}{2r_b r_c} - 1 \right)} \\ &= \sum_{cyc} \sqrt{\frac{1}{2} \left(\frac{2s^2 - (n_b - n_c)^2}{2r_b r_c} - 1 \right)} \leq \sum_{cyc} \sqrt{\frac{1}{2} \left(\frac{s^2 - s(s-a)}{s(s-a)} \right)} \\ &= \sum_{cyc} \sqrt{\frac{1}{2} \cdot \frac{4R \cos \frac{A}{2} \sin \frac{A}{2}}{4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}} = \sum_{cyc} \sqrt{\frac{2R}{r} \cdot \sin^2 \frac{A}{2}} \\ &= \sqrt{\frac{2R}{r}} \cdot \sqrt{\sum_{cyc} \sin^2 \frac{A}{2} + 2 \sum_{cyc} \left(\sin \frac{B}{2} \sin \frac{C}{2} \right)} = \sqrt{\frac{2R}{r}} \cdot \sqrt{\frac{2R-r}{2R} + \frac{1}{2R} \cdot \sum_{cyc} \frac{r}{\sin \frac{A}{2}}} \\ &= \sqrt{\frac{2R-r}{r} + \frac{1}{r} \sum_{cyc} AI} \therefore \sum_{cyc} \sqrt{\frac{1}{2} \left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1 \right)} \leq \sqrt{\frac{2R-r}{r} + \frac{1}{r} \sum_{cyc} AI} \rightarrow (1) \end{aligned}$$

Now, $\max(r_a, r_b, r_c) \geq \sqrt{4R^2 - 7r^2}$
 (Reference : Inequality in Triangle by Dang Ngoc Minh – 329;) $\stackrel{?}{\geq} 2R - r$
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$$\begin{aligned} \Leftrightarrow 4R^2 - 7r^2 &\stackrel{?}{\geq} (2R - r)^2 = 4R^2 - 4Rr + r^2 \Leftrightarrow 4Rr \stackrel{?}{\geq} 8r^2 \rightarrow \text{true via Euler} \\ \therefore \frac{2R-r}{r} &\leq \frac{\max(r_a, r_b, r_c)}{r} \rightarrow (2) \therefore (1) \text{ and } (2) \Rightarrow \end{aligned}$$

$$\sum_{cyc} \sqrt{\frac{1}{2} \left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} - 1 \right)} \leq \sqrt{\frac{\max(r_a, r_b, r_c) + AI + BI + CI}{r}} \quad \forall ABC,$$

" = " iff ΔABC is equilateral (QED)