

ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC holds :

$$\sum_{cyc} \frac{h_b h_c}{h_a} \leq \sum_{cyc} m_a \leq \frac{3}{2} \cdot \sqrt{a^2 + b^2 + c^2} \leq \sum_{cyc} \frac{g_a h_a}{r_a} \leq \sum_{cyc} \frac{w_a h_a}{r_a}$$

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$$\begin{aligned} & \left(\sum_{cyc} \frac{g_a h_a}{r_a} \right)^2 \stackrel{\text{Bogdan Fuștei}}{=} \sum_{cyc} \left(\frac{(s-a)^4 + 2rh_a(s-a)^2}{r^2 s^2} \cdot \frac{b^2 c^2}{4R^2} \right) + \\ & 2 \sum_{cyc} \left(\frac{h_b h_c}{r_b r_c} \cdot \sqrt{((s-b)^2 + 2rh_b)((s-c)^2 + 2rh_c)} \right) \rightarrow (*) \\ & \bullet \sum_{cyc} ((s-a)^4 b^2 c^2) = s^4((s^2 + 4Rr + r^2)^2 - 16Rrs^2) - 16Rrs^4(s^2 + 4Rr + r^2) \\ & + 288R^2 r^2 s^4 - 128R^2 r^2 s^4 + 32R^2 r^2 s^2(s^2 - 4Rr - r^2) \rightarrow (**) \\ & \bullet 2r \sum_{cyc} (h_a(s-a)^2 b^2 c^2) = \\ & \frac{r}{R} \left(s^2((s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2)) - \right. \\ & \left. 8Rrs^2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) + 16R^2 r^2 s^2(s^2 + 4Rr + r^2) \right) \rightarrow (***) \\ & \text{Now, } 2 \sum_{cyc} \left(\frac{h_b h_c}{r_b r_c} \cdot \sqrt{((s-b)^2 + 2rh_b)((s-c)^2 + 2rh_c)} \right) \stackrel{\text{Reverse CBS}}{\geq} \\ & \frac{4r}{R} \cdot \frac{2R}{r} \cdot \sum_{cyc} \left(\sin^2 \frac{A}{2} \left((s-b)(s-c) + \frac{2r}{2R} a \cdot \sqrt{bc} \right) \right) \\ & \stackrel{\text{GM-HM}}{\geq} \frac{8}{4Rrs} \cdot \sum_{cyc} (a(s-b)^2(s-c)^2) + \frac{8r}{R} \cdot \sum_{cyc} \left(\frac{(s-b)(s-c)}{bc} \cdot \frac{2abc}{b+c} \right) \\ & = \frac{2}{Rrs} \cdot \left(4Rrs(s^2 + 4Rr + r^2) - 32Rrs^3 + s^2 \cdot 2s(s^2 - 2Rr + r^2) + 48Rrs^3 - \right. \\ & \quad \left. 4s^3(s^2 + 4Rr + r^2) + 2s^5 \right. \\ & \quad \left. - s^2(2s(s^2 - 6Rr - 3r^2) + 2s(s^2 + 4Rr + r^2)) + \right. \\ & \quad \left. s(2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) - 16r^2 s^2 + 2(s^4 - (4Rr + r^2)^2)) \right) + \\ & \quad \frac{16r}{R} \cdot \frac{4Rrs(5s^2 + 4Rr + r^2)}{2s(s^2 + 2Rr + r^2)} \\ & = \frac{4r(8R^2 + 2Rr - s^2)}{R} + \frac{16r}{R} \cdot \frac{4rs((2R - 2r)s^2 + Rr(4R + r))}{2s(s^2 + 2Rr + r^2)} \\ & \therefore 2 \sum_{cyc} \left(\frac{h_b h_c}{r_b r_c} \cdot \sqrt{((s-b)^2 + 2rh_b)((s-c)^2 + 2rh_c)} \right) \geq \end{aligned}$$

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$$\frac{4r(8R^2 + 2Rr - s^2)}{R} + \frac{32r^2((2R - 2r)s^2 + Rr(4R + r))}{R(s^2 + 2Rr + r^2)} \rightarrow (****)$$

$$\text{So, (*) combined with (**) + (***) + (****)} \Rightarrow \left(\sum_{\text{cyc}} \frac{g_a h_a}{r_a} \right)^2 \stackrel{?}{\geq}$$

$$\frac{9}{4} \cdot 2(s^2 - 4Rr - r^2) \Leftrightarrow (R + r)s^8 - r(22R^2 + 15Rr - 4r^2)s^6 +$$

$$r^2(78R^3 - 4R^2r - 27Rr^2 + 6r^3)s^4 +$$

$$r^3(324R^4 + 512R^3r - 230R^2r^2 - 5Rr^3 + 4r^4)s^2 +$$

$$r^4(144R^5 + 620R^4r + 146R^3r^2 + 8R^2r^3 + 6Rr^4 + r^5) \stackrel{?}{\geq} 0 \text{ and } \therefore P =$$

$$(R + r)(s^2 - 16Rr + 5r^2)^4 + r(42R^2 + 29Rr - 16r^2)(s^2 - 16Rr + 5r^2)^3$$

Gerretsen

$$\stackrel{?}{\geq} 0 \therefore \text{in order to prove (1), it suffices to prove : LHS of (1) } \stackrel{?}{\geq} P$$

$$\Leftrightarrow (279R^3 + 91R^2r - 210Rr^2 + 48r^3)s^4 -$$

$$r(7774R^4 + 288R^3r - 6134R^2r^2 + 2780Rr^3 - 352r^4)s^2 +$$

$$r^2(53320R^5 - 12746R^4r - 41415R^3r^2 + 30299R^2r^3 - 7722Rr^4 + 688r^5) \stackrel{?}{\geq} 0 \text{ \&}$$

$$\therefore Q = (279R^3 + 91R^2r - 210Rr^2 + 48r^3)(s^2 - 16Rr + 5r^2)^2 +$$

$$2r(577R^4 - 83R^3r - 748R^2r^2 + 428Rr^3 - 64r^4)(s^2 - 16Rr + 5r^2) \stackrel{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove (2), it suffices to prove : LHS of (2) } \stackrel{?}{\geq} Q$$

$$\Leftrightarrow 90t^5 + 43t^4 - 794t^3 + 828t^2 - 280t + 32 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(90t^4 + 49t^3 + 174t^2(t - 2) + 132t - 16) \stackrel{?}{\geq} 0 \rightarrow \text{true} \therefore t \stackrel{\text{Euler}}{\geq} 2$$

$$\Rightarrow (2) \Rightarrow (1) \text{ is true} \therefore \frac{3}{2} \cdot \sqrt{a^2 + b^2 + c^2} \leq \sum_{\text{cyc}} \frac{g_a h_a}{r_a}; \text{ Again, triangle inequality} \Rightarrow$$

$$g_a \leq AI + r \stackrel{?}{\leq} w_a \Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{8Rrs \cos \frac{A}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2} (b + c)} \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a + b + c}{(b + c) \sin \frac{A}{2}}$$

$$\Leftrightarrow (b + c) \sin \frac{A}{2} \stackrel{?}{\leq} a \Leftrightarrow \cos \frac{B - C}{2} \stackrel{?}{\leq} 1 \rightarrow \text{true} \therefore g_a \leq w_a \text{ and analogs}$$

$$\therefore \sum_{\text{cyc}} \frac{g_a h_a}{r_a} \leq \sum_{\text{cyc}} \frac{w_a h_a}{r_a} \text{ and also, } \sum_{\text{cyc}} m_a \stackrel{\text{CBS}}{\leq} \sqrt{3 \cdot \frac{3}{4} \sum_{\text{cyc}} a^2} = \frac{3}{2} \cdot \sqrt{\sum_{\text{cyc}} a^2} \text{ and}$$

$$\sum_{\text{cyc}} \frac{h_b h_c}{h_a} \leq \sum_{\text{cyc}} m_a \left(\text{Reference : Inequality in Triangle by Dang Ngoc Minh} \right)$$

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$\forall \Delta ABC, '' = ''$ iff ΔABC is equilateral (QED)