

# ROMANIAN MATHEMATICAL MAGAZINE

In any  $\Delta ABC$  the following relationship holds :

$$m_a \cdot \sqrt{\frac{r_a}{h_a}} + m_b \cdot \sqrt{\frac{r_b}{h_b}} + m_c \cdot \sqrt{\frac{r_c}{h_c}} \geq \frac{(n_a + n_b + n_c)(h_a + h_b + h_c)}{r_a + r_b + r_c}$$

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$$\begin{aligned} \frac{\sum_{cyc} n_a}{4R+r} &\stackrel{CBS}{\leq} \frac{\sqrt{3 \sum_{cyc} n_a^2}}{4R+r} \stackrel{Bogdan Fustei}{=} \frac{\sqrt{3 \sum_{cyc} \left(s^2 - \frac{r}{R} \cdot s^2 \cdot \sec^2 \frac{A}{2}\right)}}{4R+r} \\ &= \frac{\sqrt{3 \left(\frac{(3R-r)s^2 - r(4R+r)^2}{R}\right)}}{4R+r} \stackrel{?}{\leq} \sqrt{\frac{R}{2r}} \stackrel{squaring}{\Leftrightarrow} R^2(4R+r)^2 + 6r^2(4R+r)^2 \stackrel{?}{\geq} 6r(3R-r)s^2 \end{aligned}$$

$$\begin{aligned} \text{Now, } 6r(3R-r)s^2 &\stackrel{Rouche}{\leq} 6r(3R-r) \left(2R^2 + 10Rr - r^2 + 2(R-2r) \cdot \sqrt{R^2 - 2Rr}\right) \\ &\stackrel{?}{\leq} (R^2 + 6r^2)(4R+r)^2 \end{aligned}$$

$$\Leftrightarrow R(R-2r)(16R^2 + 4Rr - 63r^2) \stackrel{?}{\geq} 6r(3R-r) \cdot 2(R-2r) \cdot \sqrt{R^2 - 2Rr}$$

and  $\because R-2r \stackrel{Euler}{\geq} 0 \therefore$  it suffices to prove following squaring :

$$\begin{aligned} R^2(16R^2 + 4Rr - 63r^2)^2 &\stackrel{?}{\geq} 144r^2(3R-r)^2(R^2 - 2Rr) \\ \Leftrightarrow 256t^5 + 128t^4 - 3296t^3 + 2952t^2 + 2097t + 288 &\geq 0 \left(t = \frac{R}{r}\right) \\ \Leftrightarrow \frac{(4t+1)^2}{100} \left( \left(16t + \frac{336}{5}\right)(10t-21)^2 + \frac{2340(t-2) + 504}{5} \right) &\stackrel{?}{\geq} 0 \rightarrow \text{true} \end{aligned}$$

(strict inequality)  $\because t \stackrel{Euler}{\geq} 2 \Rightarrow (*)$  is true  $\Rightarrow \frac{\sum_{cyc} n_a}{\sum_{cyc} r_a} \leq \sqrt{\frac{R}{2r}} \rightarrow \textcircled{1}$  and now,

$$m_a \cdot \sqrt{\frac{r_a}{h_a}} \stackrel{Lascu}{\geq} \frac{b+c}{2} \cdot \cos \frac{A}{2} \cdot \sqrt{\frac{s \tan \frac{A}{2} \cdot 4R \cos \frac{A}{2} \sin \frac{A}{2}}{2rs}} = \sqrt{\frac{R}{2r}} \cdot \frac{(b+c)a}{4R} \text{ and analogs}$$

$$\therefore \sum_{cyc} \left( m_a \cdot \sqrt{\frac{r_a}{h_a}} \right) \geq \sqrt{\frac{R}{2r}} \cdot \sum_{cyc} \frac{(b+c)a}{4R} \stackrel{via \textcircled{1}}{\geq} \frac{\sum_{cyc} n_a}{\sum_{cyc} r_a} \cdot \sum_{cyc} h_a \text{ and so,}$$

$$m_a \cdot \sqrt{\frac{r_a}{h_a}} + m_b \cdot \sqrt{\frac{r_b}{h_b}} + m_c \cdot \sqrt{\frac{r_c}{h_c}} \geq \frac{(n_a + n_b + n_c)(h_a + h_b + h_c)}{r_a + r_b + r_c} \quad \forall ABC,$$

" = " iff  $\Delta ABC$  is equilateral (QED)