

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\sqrt{\frac{R}{2r}} \cdot \left(3s - \sum_{cyc} \left(p_a \cdot \sqrt{\frac{l_a}{g_a}} \right) \right) \geq 2 \sum_{cyc} \frac{g_b g_c}{n_a + s}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} m_a n_a &\stackrel{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \stackrel{\text{Fustei and Ajiba}}{\Leftrightarrow} \\ &\left(s(s-a) + \frac{(b-c)^2}{4} \right) \left(s(s-a) + \frac{s(b-c)^2}{a} \right) \stackrel{?}{\geq} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \right)^2 \\ &\quad + \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9} \cdot \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \right) \\ &\Leftrightarrow s(s-a)(b-c)^2 \left(\frac{s}{a} + \frac{1}{4} \right) + \frac{s(b-c)^4}{4a} \stackrel{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} + \\ &\quad 2s(s-a) \cdot \frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a) \cdot \frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2} \\ &\Leftrightarrow s(s-a) \left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9} \right) + \\ &\quad \left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2} \right) (b-c)^2 \stackrel{?}{\geq} 0 \quad (\because (b-c)^2 \geq 0) \\ &\Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} + \\ &\quad \frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4} \cdot (b-c)^2 \stackrel{?}{\geq} 0 \\ &\Leftrightarrow \frac{s(s-a) \left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3 \right)}{36a(2s+a)^2} + \\ &\quad \frac{(s-a) \left((s-a)(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3) + 1701a^4 \right)}{324a(2s+a)^4} \cdot (b-c)^2 \\ &\stackrel{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \rightarrow \textcircled{1} \end{aligned}$$

$$\begin{aligned} &\text{and } n_a g_a \stackrel{\text{Bogdan Fustei}}{\geq} m_a l_a \rightarrow \textcircled{2} \therefore \textcircled{1} \cdot \textcircled{2} \Rightarrow (m_a n_a)(n_a g_a) \geq p_a^2 \cdot m_a l_a \\ &\Rightarrow n_a \geq p_a \cdot \sqrt{\frac{l_a}{g_a}} \Rightarrow \sqrt{\frac{R}{2r}} \cdot \left(s - p_a \cdot \sqrt{\frac{l_a}{g_a}} \right) (s + n_a) \geq \sqrt{\frac{R}{2r}} \cdot (s^2 - n_a^2) \stackrel{\text{Bogdan Fustei}}{=} \\ &\quad \sqrt{\frac{R}{2r}} \cdot \frac{2h_a r_a w_a}{w_a} \geq \sqrt{\frac{R}{2r}} \cdot \frac{2h_a g_b g_c}{w_a} \end{aligned}$$

(Reference : Solution to Inequality in Triangle – 127)
by Bogdan Fustei; published at www.ssmrmh.ro)

$$\therefore \sqrt{\frac{R}{2r}} \cdot \left(s - p_a \cdot \sqrt{\frac{l_a}{g_a}} \right) (s + n_a) \stackrel{(*)}{\geq} \sqrt{\frac{R}{2r}} \cdot \frac{2h_a g_b g_c}{w_a}$$

$$\begin{aligned} \text{Now, } (b+c)^2 &\stackrel{?}{\geq} 32Rr \cdot \cos^2 \frac{A}{2} = 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s-b} + \frac{1}{s-c} \right) \\ &= 8(s-a)(s-b)(s-c) \cdot \frac{a}{(s-b)(s-c)} = 4a(b+c-a) \end{aligned}$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \stackrel{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true}$$

$$\therefore b+c \geq \sqrt{32Rr} \cdot \cos \frac{A}{2} \text{ and analogs} \Rightarrow \frac{w_a}{h_a} = \frac{2R \cdot 2bc \cdot \cos \frac{A}{2}}{(b+c)bc} \leq \frac{4R \cdot \cos \frac{A}{2}}{\sqrt{32Rr} \cdot \cos \frac{A}{2}} = \sqrt{\frac{R}{2r}}$$

$$\Rightarrow \sqrt{\frac{R}{2r}} \cdot \frac{2h_a g_b g_c}{w_a} \geq 2 \cdot \sqrt{\frac{R}{2r}} \cdot \frac{g_b g_c}{\sqrt{\frac{R}{2r}}} \Rightarrow \sqrt{\frac{R}{2r}} \cdot \frac{2h_a g_b g_c}{w_a} \stackrel{(**)}{\geq} 2g_b g_c \therefore (*) \text{ and } (**) \Rightarrow$$

$$\sqrt{\frac{R}{2r}} \cdot \left(s - p_a \cdot \sqrt{\frac{l_a}{g_a}} \right) \geq \frac{2g_b g_c}{s + n_a} \text{ and analogs and so,}$$

$$\sqrt{\frac{R}{2r}} \cdot \left(3s - \sum_{\text{cyc}} \left(p_a \cdot \sqrt{\frac{l_a}{g_a}} \right) \right) \geq 2 \sum_{\text{cyc}} \frac{g_b g_c}{n_a + s} \quad \forall \triangle ABC,$$

" = " iff $\triangle ABC$ is equilateral (QED)