ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{\frac{R}{2r}}\cdot(r_a+r_b+r_c)\geq\frac{g_bg_c}{h_a}+\frac{g_cg_a}{h_b}+\frac{g_ag_b}{h_c}$$

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Solution by Soumava Chakraborty-Kolkata-India

Let $x=s-a, y=s-b, z=s-c$ and then : $a=y+z, b=z+x,$ $c=x+y$ and $s=x+y+z$ and furthermore, we denote : $\frac{y+z}{x}=m$ and $\frac{yz}{x^2}=n$ and then, we have the following set "S" of relations : $y^2+z^2=x^2(m^2-2n)$, $y^3+z^3=x^3(m^3-3nn)$, and now, $g_bg_c\overset{?}{\leq}r_aw_a \overset{\text{squaring and via Bogdan Fustei}}{\Leftrightarrow}$

$$\left((s-b)^2+\frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2+\frac{4(s-a)(s-b)(s-c)}{c}\right)\overset{?}{\leq}\frac{s(s-b)(s-c)}{s-a}\cdot\frac{4bc}{(b+c)^2}\cdot s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2\overset{?}{\geq}(4zx+xy+yz)(4xy+zx+yz)(2x+y+z)^2$$

$$\Leftrightarrow 4x^6+16x^5(y+z)+8x^4(y^2+z^2)-12x^4.yz-32x^3.yz(y+z)-$$
$$5x^2.yz(y+z)^2+3x.yz(y^3+z^3)+13x.y^2z^2(y+z)+3y^2z^2(y+z)^2\ \boxed{\overset{?}{\underset{①}{\geq}}}\ 0$$ and via

set of relations "S", to prove ①, it suffices to prove, following simplification :

$$\overbrace{(3m^2+4m)}^{\Omega_1}n^2+\overbrace{(3m^3-5m^2-32m-28)}^{\Omega_2}n+\overbrace{(8m^2+16m+4)}^{\Omega_3}+\ \boxed{\overset{?}{\underset{②}{\geq}}}\ 0$$

and it's trivially true when : $\Omega_2\geq 0$ and when : $\Omega_2<0$, then :

$$3m^3-5m^2-32m-28=(m-5)(3m^2+10m+18)+62<0$$
$$\Rightarrow(m-5)(3m^2+10m+18)<-62<0\Rightarrow m<5\rightarrow\text{(i)}$$

Now, LHS of ② is a quadratic polynomial in "n" with discriminant, $\boxed{\delta}=$ $\sigma_2^2-4\sigma_1\sigma_3=(m+1)(m+2)^2(9m^3-75m^2+40m+196)$ and when : $9m^3-75m^2+40m+196\leq 0$, then $\delta\leq 0\Rightarrow$ ② is trivially true and when : $9m^3-75m^2+40m+196>0$, then : $9m^3-75m^2+40m+196$

$$=\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)-\frac{14}{3}>0$$

$$\Rightarrow\left(3m(m-5)-3m-\frac{86}{3}\right)(3m-7)>\frac{14}{3}>0 \text{ and } \because m\overset{\text{via (i)}}{\leq}5$$

$$\Rightarrow 3m(m-5)-3m-\frac{86}{3}<0\ \therefore m<\frac{7}{3}\rightarrow\text{(ii) and so, when : }\delta>0,$$

then it suffices to prove : $2\Omega_1.n\overset{?}{\leq}-\Omega_2-\sqrt{\delta}\Leftrightarrow$ and $\because n\overset{AM-GM}{\leq}\frac{m^2}{4}$

$$\therefore \text{ suffices to prove : } \Omega_1.\frac{m^2}{2} \overset{?}{\leq} -\Omega_2 - \sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2-2m-14) \overset{?}{\geq} 2.\sqrt{\delta}$$

$$and \because 3m^2-2m-14 = \left(m+\frac{5}{3}\right)(3m-7) - \frac{7}{3} \overset{\text{via (ii)}}{<} 0 \therefore \text{ it suffices to prove :}$$

$$(m+2)^4(3m^2-2m-14)^2 \overset{?}{\geq} 4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4) \overset{?}{\geq} 0}$$

$$\therefore g_b g_c \leq r_a w_a \text{ and analogs} \to (*) \text{ and now, } (b+c)^2 \overset{?}{\geq} 32Rr.\cos^2\frac{A}{2}$$

$$= 8r(r_b+r_c) = 8r^2 s\left(\frac{1}{s-b}+\frac{1}{s-c}\right) = 8(s-a)(s-b)(s-c).\frac{a}{(s-b)(s-c)}$$

$$= 4a(b+c-a) \Leftrightarrow (b+c)^2+4a^2-4a(b+c) \overset{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore b+c \geq \sqrt{32Rr}.\cos\frac{A}{2} \text{ and analogs} \to (**) \Rightarrow \frac{g_b g_c}{h_a} \overset{\text{via } (*)}{\leq} \frac{r_a w_a}{h_a}$$

$$= r_a.\frac{2R.2bc.\cos\frac{A}{2}}{(b+c)bc} \overset{\text{via } (**)}{\leq} r_a.\frac{4R.\cos\frac{A}{2}}{\sqrt{32Rr}.\cos\frac{A}{2}} = r_a.\sqrt{\frac{R}{2r}} \therefore \frac{g_b g_c}{h_a} \leq \sqrt{\frac{R}{2r}}.r_a \text{ and analogs}$$

$$\Rightarrow \frac{g_b g_c}{h_a} + \frac{g_c g_a}{h_b} + \frac{g_a g_b}{h_c} \leq \sqrt{\frac{R}{2r}}.(r_a+r_b+r_c)\ \forall\, \Delta\, ABC,$$

$$'' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$