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In any ΔABC the following relationship holds :

$$\sqrt{\frac{R}{2r}} \cdot (h_a + h_b + h_c) \geq \frac{g_b g_c}{r_a} + \frac{g_c g_a}{r_b} + \frac{g_a g_b}{r_c}$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x, c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$,

$$\begin{aligned} y^3 + z^3 &= x^3(m^3 - 3mn), \text{ and now, } g_b g_c \leq r_a w_a \text{ and via Bogdan Fustei} \\ &\Leftrightarrow \left((s-b)^2 + \frac{4(s-a)(s-b)(s-c)}{b} \right) \left((s-c)^2 + \frac{4(s-a)(s-b)(s-c)}{c} \right) \leq \\ &\quad \frac{s(s-b)(s-c)}{s-a} \cdot \frac{4bc}{(b+c)^2} \cdot s(s-a) \\ &\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2 \geq (4zx + xy + yz)(4xy + zx + yz)(2x + y + z)^2 \\ &\Leftrightarrow 4x^6 + 16x^5(y+z) + 8x^4(y^2+z^2) - 12x^4 \cdot yz - 32x^3 \cdot yz(y+z) - \\ &\quad 5x^2 \cdot yz(y+z)^2 + 3x \cdot yz(y^3+z^3) + 13x \cdot y^2 z^2(y+z) + 3y^2 z^2(y+z)^2 \geq 0 \text{ and via} \end{aligned}$$

set of relations "S", to prove ①, it suffices to prove, following simplification :

$$\overbrace{(3m^2 + 4m)}^{\Omega_1} n^2 + \overbrace{(3m^3 - 5m^2 - 32m - 28)}^{\Omega_2} n + \overbrace{(8m^2 + 16m + 4)}^{\Omega_3} + 0$$

and it's trivially true when : $\Omega_2 \geq 0$ and when : $\Omega_2 < 0$, then :

$$3m^3 - 5m^2 - 32m - 28 = (m-5)(3m^2 + 10m + 18) + 62 < 0$$

$$\Rightarrow (m-5)(3m^2 + 10m + 18) < -62 < 0 \Rightarrow m < 5 \rightarrow \text{(i)}$$

Now, LHS of ② is a quadratic polynomial in "n" with discriminant, $\delta =$

$$\sigma_2^2 - 4\sigma_1\sigma_3 = (m+1)(m+2)^2(9m^3 - 75m^2 + 40m + 196) \text{ and when : } 9m^3 - 75m^2 + 40m + 196 \leq 0, \text{ then } \delta \leq 0 \Rightarrow \text{② is trivially true and when :}$$

$$9m^3 - 75m^2 + 40m + 196 > 0, \text{ then : } 9m^3 - 75m^2 + 40m + 196$$

$$= \left(3m(m-5) - 3m - \frac{86}{3} \right) \left(3m-7 \right) - \frac{14}{3} > 0$$

$$\Rightarrow \left(3m(m-5) - 3m - \frac{86}{3} \right) (3m-7) > \frac{14}{3} > 0 \text{ and } \therefore m \stackrel{\text{via (i)}}{\leq} 5$$

$$\Rightarrow 3m(m-5) - 3m - \frac{86}{3} < 0 \therefore m < \frac{7}{3} \rightarrow \text{(ii) and so, when : } \delta > 0,$$

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then it suffices to prove : $2\Omega_1 \cdot n \stackrel{?}{\leq} -\Omega_2 - \sqrt{\delta} \Leftrightarrow$ and $\because n \stackrel{\text{AM-GM}}{\leq} \frac{m^2}{4}$

$\therefore$ suffices to prove : $\Omega_1 \cdot \frac{m^2}{2} \stackrel{?}{\leq} -\Omega_2 - \sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2-2m-14) \stackrel{?}{\geq} 2\sqrt{\delta}$

and $\because 3m^2-2m-14 = \left(m+\frac{5}{3}\right)(3m-7) - \frac{7}{3} \stackrel{\text{via (ii)}}{<} 0 \therefore$ it suffices to prove :

$$(m+2)^4(3m^2-2m-14)^2 \stackrel{?}{\geq} 4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4) \stackrel{?}{\geq} 0}$$

$\therefore g_b g_c \leq r_a w_a$ and analogs $\rightarrow (*)$ and now, $(b+c)^2 \stackrel{?}{\geq} 32Rr \cdot \cos^2 \frac{A}{2}$

$$= 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s-b} + \frac{1}{s-c} \right) = 8(s-a)(s-b)(s-c) \cdot \frac{a}{(s-b)(s-c)}$$

$$= 4a(b+c-a) \Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \stackrel{?}{\geq} 0 \Leftrightarrow (b+c-2a)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true}$$

$\therefore b+c \geq \sqrt{32Rr \cdot \cos^2 \frac{A}{2}}$ and analogs $\rightarrow (**)$ $\Rightarrow \frac{g_b g_c}{r_a} \stackrel{\text{via } (*)}{\leq} \frac{r_a w_a}{r_a} = h_a \cdot \frac{w_a}{h_a}$

$$= h_a \cdot \frac{2R \cdot 2bc \cdot \cos^2 \frac{A}{2}}{(b+c)bc} \stackrel{\text{via } (**)}{\leq} h_a \cdot \frac{4R \cdot \cos^2 \frac{A}{2}}{\sqrt{32Rr \cdot \cos^2 \frac{A}{2}}} = h_a \cdot \sqrt{\frac{R}{2r}}$$

$\therefore \frac{g_b g_c}{r_a} \leq \sqrt{\frac{R}{2r}} \cdot h_a$ and analogs $\Rightarrow \frac{g_b g_c}{r_a} + \frac{g_c g_a}{r_b} + \frac{g_a g_b}{r_c} \leq \sqrt{\frac{R}{2r}} \cdot (h_a + h_b + h_c)$

$\forall \Delta ABC, '' = ''$ iff ΔABC is equilateral (QED)