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In any Δ ABC the following relationship holds :

$$\frac{n_a}{h_a} \geq \frac{1}{2}\left(\frac{g_c}{l_b} + \frac{g_b}{l_c}\right)$$

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Let $x = s - a, y = s - b, z = s - c$ and then : $a = y + z, b = z + x,$
$c = x + y$ and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$
and then, we have the following set $''S''$ of relations : $y^2 + z^2 = x^2(m^2 - 2n),$

$$y^3 + z^3 = x^3(m^3 - 3nn), \text{ and now, } g_b g_c \overset{?}{\leq} r_a w_a \overset{\text{squaring and via Bogdan Fustei}}{\Leftrightarrow}$$

$$\left((s-b)^2 + \frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2 + \frac{4(s-a)(s-b)(s-c)}{c}\right) \overset{?}{\leq} \frac{s(s-b)(s-c)}{s-a}.\frac{4bc}{(b+c)^2}.s(s-a)$$

$$\Leftrightarrow 4(z+x)^2(x+y)^2(x+y+z)^2 \overset{?}{\geq} (4zx + xy + yz)(4xy + zx + yz)(2x + y + z)^2$$

$$\Leftrightarrow 4x^6 + 16x^5(y+z) + 8x^4(y^2+z^2) - 12x^4.yz - 32x^3.yz(y+z) -$$

$$5x^2.yz(y+z)^2 + 3x.yz(y^3+z^3) + 13x.y^2z^2(y+z) + 3y^2z^2(y+z)^2 \boxed{\underset{①}{\overset{?}{\geq}}} 0 \text{ and via}$$

set of relations $''S''$, to prove ①, it suffices to prove, following simplification :

$$\overbrace{(3m^2 + 4m)}^{\Omega_1} n^2 + \overbrace{(3m^3 - 5m^2 - 32m - 28)}^{\Omega_2} n + \overbrace{(8m^2 + 16m + 4)}^{\Omega_3} + \boxed{\underset{②}{\overset{?}{\geq}}} 0$$

and it's trivially true when : $\Omega_2 \geq 0$ and when : $\Omega_2 < 0$, then :

$$3m^3 - 5m^2 - 32m - 28 = (m-5)(3m^2 + 10m + 18) + 62 < 0$$

$$\Rightarrow (m-5)(3m^2 + 10m + 18) < -62 < 0 \Rightarrow m < 5 \rightarrow \text{(i)}$$

Now, LHS of ② is a quadratic polynomial in $''n''$ with discriminant, $\boxed{\delta} =$ $\sigma_2^2 - 4\sigma_1\sigma_3 = (m+1)(m+2)^2(9m^3 - 75m^2 + 40m + 196)$ and when : $9m^3 - 75m^2 + 40m + 196 \leq 0$, then $\delta \leq 0 \Rightarrow$ ② is trivially true and when : $9m^3 - 75m^2 + 40m + 196 > 0$, then : $9m^3 - 75m^2 + 40m + 196$

$$= \left(3m(m-5) - 3m - \frac{86}{3}\right)(3m - 7) - \frac{14}{3} > 0$$

$$\Rightarrow \left(3m(m-5) - 3m - \frac{86}{3}\right)(3m - 7) > \frac{14}{3} > 0 \text{ and } \because m \overset{\text{via (i)}}{\leq} 5$$

$$\Rightarrow 3m(m-5) - 3m - \frac{86}{3} < 0 \therefore m < \frac{7}{3} \rightarrow \text{(ii) and so, when : } \delta > 0,$$

then it suffices to prove : $2\Omega_1.n \overset{?}{\leq} -\Omega_2 - \sqrt{\delta} \Leftrightarrow$ and $\because n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4}$

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$$\therefore \text{ suffices to prove : } \Omega_1 . \frac{m^2}{2} \overset{?}{\leq} -\Omega_2 - \sqrt{\delta} \Leftrightarrow -(m+2)^2(3m^2-2m-14) \overset{?}{\geq} 2.\sqrt{\delta}$$

$$\text{and} \because 3m^2-2m-14 = \left(m+\frac{5}{3}\right)(3m-7) - \frac{7}{3} \overset{\text{via (ii)}}{<} 0 \therefore \text{ it suffices to prove :}$$

$$(m+2)^4(3m^2-2m-14)^2 \overset{?}{\geq} 4(m+1)(m+2)^2(9m^3-75m^2+40m+196)$$

$$\Leftrightarrow \boxed{m(m+2)^2(m-2)^2(3m+4)(3m^2+16m+4) \overset{?}{\geq} 0} \therefore g_b g_c \leq r_a w_a$$

$$\text{and analogously, } \boxed{\boxed{g_c g_a \leq r_b l_b}} \to (*) \text{ and } \boxed{\boxed{g_a g_b \leq r_c l_c}} \to (**)$$

$$\text{Now, } n_a \geq \frac{m_a l_a}{g_a} \Rightarrow \frac{2n_a}{h_a} \overset{\text{Bogdan Fustei}}{\geq} \frac{2m_a l_a}{h_a g_a} \overset{\text{Lascu + AM-GM}}{\geq} 4R.\frac{s(s-a)}{bc}.\frac{1}{g_a}$$

$$= 4R\cos^2\frac{A}{2}.\frac{1}{g_a} = \frac{r_b+r_c}{g_a} \overset{?}{\geq} \frac{g_c}{l_b} + \frac{g_b}{l_c} \Leftrightarrow r_b + r_c \overset{?}{\geq} \frac{g_c g_a}{l_b} + \frac{g_a g_b}{l_c} \to \text{true}$$

$$\because \frac{g_c g_a}{l_b} + \frac{g_a g_b}{l_c} \overset{\text{via } (*) \text{ and } (**)}{\leq} \frac{r_b l_b}{l_b} + \frac{r_c l_c}{l_c} = r_b + r_c \therefore \frac{n_a}{h_a} \geq \frac{1}{2}\left(\frac{g_c}{l_b} + \frac{g_b}{l_c}\right) \forall \Delta ABC,$$

$$'' = '' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$