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In any ΔABC the following relationship holds :

$$\sqrt{\frac{2R}{r}} \cdot \left(2R + \frac{h_a + h_b + h_c}{2} - r \right) \geq 2(n_a + n_b + n_c) - \sum_{cyc} \sqrt{(b-c)^2 + r^2}$$

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$$\begin{aligned} \frac{\sum_{cyc} n_a}{4R+r} &\stackrel{CBS}{\leq} \frac{\sqrt{3 \sum_{cyc} n_a^2}}{4R+r} \stackrel{Bogdan Fustei}{=} \frac{\sqrt{3 \sum_{cyc} \left(s^2 - \frac{r}{R} \cdot s^2 \cdot \sec^2 \frac{A}{2} \right)}}{4R+r} \\ &= \frac{\sqrt{3 \left(\frac{(3R-r)s^2 - r(4R+r)^2}{R} \right)}}{4R+r} \stackrel{?}{\leq} \sqrt{\frac{R}{2r}} \quad \Leftrightarrow \quad \text{squaring} \end{aligned}$$

$$R^2(4R+r)^2 + 6r^2(4R+r)^2 \stackrel{?}{\geq} 6r(3R-r)s^2 \quad (*)$$

$$\begin{aligned} \text{Now, } 6r(3R-r)s^2 &\stackrel{Rouche}{\leq} 6r(3R-r) \left(2R^2 + 10Rr - r^2 + 2(R-2r) \cdot \sqrt{R^2 - 2Rr} \right) \\ &\stackrel{?}{\leq} (R^2 + 6r^2)(4R+r)^2 \end{aligned}$$

$$\Leftrightarrow R(R-2r)(16R^2 + 4Rr - 63r^2) \stackrel{?}{\geq} 6r(3R-r) \cdot 2(R-2r) \cdot \sqrt{R^2 - 2Rr}$$

and $\because R-2r \stackrel{Euler}{\geq} 0 \therefore$ it suffices to prove following squaring :

$$R^2(16R^2 + 4Rr - 63r^2)^2 \stackrel{?}{\geq} 144r^2(3R-r)^2(R^2 - 2Rr)$$

$$\Leftrightarrow 256t^5 + 128t^4 - 3296t^3 + 2952t^2 + 2097t + 288 \stackrel{?}{\geq} 0 \quad \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow \frac{(4t+1)^2}{100} \left(\left(16t + \frac{336}{5} \right) (10t-21)^2 + \frac{2340(t-2) + 504}{5} \right) \stackrel{?}{\geq} 0$$

$\rightarrow$ true (strict inequality) $\because t \stackrel{Euler}{\geq} 2 \Rightarrow (*)$ is true $\therefore \sum_{cyc} n_a \leq \sqrt{\frac{R}{2r}} \cdot (4R+r) \rightarrow \textcircled{1}$

$$\text{and also, } \sum_{cyc} n_a \leq \sum_{cyc} AI + \sum_{cyc} \sqrt{(b-c)^2 + r^2} \rightarrow \textcircled{2}$$

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(Reference : Inequality in Triangle by Bogdan Fustei – 103;) $\therefore$ ① + ② $\Rightarrow$
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$$2 \sum_{\text{cyc}} n_a - \sum_{\text{cyc}} \sqrt{(b-c)^2 + r^2} \leq \sqrt{\frac{R}{2r}} \cdot (4R + r) + \sum_{\text{cyc}} AI \stackrel{?}{\leq}$$

$$\sqrt{\frac{2R}{r}} \cdot \left(2R + \frac{h_a + h_b + h_c}{2} - r \right) = \sqrt{\frac{R}{2r}} \cdot \left(4R - 2r + \frac{s^2 + 4Rr + r^2}{2R} \right)$$

$$\Leftrightarrow \sum_{\text{cyc}} AI \stackrel{?}{\geq} \sqrt{\frac{R}{2r}} \cdot \left(\frac{s^2 - 2Rr + r^2}{2R} \right) \text{ and we have :}$$

$$\sum_{\text{cyc}} AI = r \cdot \sum_{\text{cyc}} \sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}} \stackrel{\text{CBS}}{\leq} \frac{r}{r \cdot \sqrt{s}} \cdot \sqrt{\sum_{\text{cyc}} ab} \cdot \sqrt{\sum_{\text{cyc}} (s-a)}$$

$$= \sqrt{s^2 + 4Rr + r^2} \stackrel{?}{\leq} \sqrt{\frac{R}{2r}} \cdot \left(\frac{s^2 - 2Rr + r^2}{2R} \right)$$

$$\Leftrightarrow (s^2 - 2Rr + r^2)^2 \stackrel{?}{\geq} 8Rr(s^2 + 4Rr + r^2)$$

$$\Leftrightarrow s^4 - (12Rr - 2r^2)s^2 - r^2(28R^2 + 12Rr - r^2) \stackrel{?}{\geq} 0 \text{ and via Gerretsen,}$$

$$\text{LHS of } (\bullet) \geq (4Rr - 3r^2)s^2 - r^2(28R^2 + 12Rr - r^2) \stackrel{\text{Gerretsen}}{\geq}$$

$$(4Rr - 3r^2)(16Rr - 5r^2) - r^2(28R^2 + 12Rr - r^2) = 4r^2(R - 2r)(9R - 2r) \stackrel{\text{Euler}}{\geq} 0$$

$\Rightarrow (\bullet) \Rightarrow (\blacksquare)$ is true and so,

$$\sqrt{\frac{2R}{r}} \cdot \left(2R + \frac{h_a + h_b + h_c}{2} - r \right) \geq 2(n_a + n_b + n_c) - \sum_{\text{cyc}} \sqrt{(b-c)^2 + r^2} \quad \forall ABC,$$

" = " iff $\triangle ABC$ is equilateral (QED)