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In any Δ ABC the following relationship holds :

$$\sqrt{\frac{2R}{r}}\cdot\left(max(r_a,r_b,r_c)+\frac{h_a+h_b+h_c}{2}\right)\ge 2(n_a+n_b+n_c)-\sum_{cyc}\sqrt{(b-c)^2+r^2}$$

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$$\frac{\sum_{cyc}n_a}{4R+r}\overset{CBS}{\le}\frac{\sqrt{3\sum_{cyc}n_a^2}}{4R+r}\overset{Bogdan\ Fustei}{=}\frac{\sqrt{3\sum_{cyc}\left(s^2-\frac{r}{R}.s^2.\sec^2\frac{A}{2}\right)}}{4R+r}$$

$$=\frac{\sqrt{3\left(\frac{(3R-r)s^2-r(4R+r)^2}{R}\right)}}{4R+r}\overset{?}{\le}\sqrt{\frac{R}{2r}}\overset{squaring}{\Leftrightarrow}$$

$$R^2(4R+r)^2+6r^2(4R+r)^2\ \boxed{\overset{?}{\underset{(*)}{\ge}}}\ 6r(3R-r)s^2$$

$$\text{Now},6r(3R-r)s^2\overset{Rouche}{\le}6r(3R-r)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\le}(R^2+6r^2)(4R+r)^2$$

$$\Leftrightarrow R(R-2r)(16R^2+4Rr-63r^2)\overset{?}{\ge}6r(3R-r).2(R-2r).\sqrt{R^2-2Rr}$$

$$and\ \because R-2r\overset{Euler}{\ge}0\ \therefore\ \text{it suffices to prove following squaring :}$$

$$R^2(16R^2+4Rr-63r^2)^2\overset{?}{\ge}144r^2(3R-r)^2(R^2-2Rr)$$

$$\Leftrightarrow 256t^5+128t^4-3296t^3+2952t^2+2097t+288\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow\frac{(4t+1)^2}{100}\left(\left(16t+\frac{336}{5}\right)(10t-21)^2+\frac{2340(t-2)+504}{5}\right)\overset{?}{\ge}0$$

$$\to \text{true (strict inequality)}\because t\overset{Euler}{\ge}2\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}n_a\le\sqrt{\frac{R}{2r}}.(4R+r)\to ①$$

$$and\ also,\sum_{cyc}n_a\le\sum_{cyc}AI+\sum_{cyc}\sqrt{(b-c)^2+r^2}\to ②$$

(Reference : Inequality in Triangle by Bogdan Fustei – 103; published at www.ssmrmh.ro) $\therefore$ ① + ② $\Rightarrow$

$$2\sum_{cyc}n_a-\sum_{cyc}\sqrt{(b-c)^2+r^2}\le\sqrt{\frac{R}{2r}}.(4R+r)+\sum_{cyc}AI\overset{?}{\le}$$

$$\sqrt{\frac{2R}{r}}\cdot\left(2R+\frac{h_a+h_b+h_c}{2}-r\right)=\sqrt{\frac{R}{2r}}\cdot\left(4R-2r+\frac{s^2+4Rr+r^2}{2R}\right)$$

$$\Leftrightarrow \sum_{cyc} AI \boxed{\overset{?}{\underset{(\blacksquare)}{\leq}}} \sqrt{\frac{R}{2r}}\cdot\left(\frac{s^2-2Rr+r^2}{2R}\right) \text{ and we have :}$$

$$\sum_{cyc} AI = r.\sum_{cyc}\sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}} \overset{CBS}{\leq} \frac{r}{r.\sqrt{s}}\cdot\sqrt{\sum_{cyc} ab}\,.\sqrt{\sum_{cyc}(s-a)}$$

$$=\sqrt{s^2+4Rr+r^2} \overset{?}{\leq} \sqrt{\frac{R}{2r}}\cdot\left(\frac{s^2-2Rr+r^2}{2R}\right)$$

$$\Leftrightarrow (s^2-2Rr+r^2)^2 \overset{?}{\geq} 8Rr(s^2+4Rr+r^2)$$

$$\Leftrightarrow s^4-(12Rr-2r^2)s^2-r^2(28R^2+12Rr-r^2) \boxed{\overset{?}{\underset{(\bullet)}{\geq}}} 0 \text{ and via Gerretsen,}$$

$$\text{LHS of } (\bullet) \geq (4Rr-3r^2)s^2-r^2(28R^2+12Rr-r^2) \overset{Gerretsen}{\geq}$$

$$(4Rr-3r^2)(16Rr-5r^2)-r^2(28R^2+12Rr-r^2)=4r^2(R-2r)(9R-2r) \overset{Euler}{\geq} 0$$

$$\Rightarrow (\bullet) \Rightarrow (\blacksquare) \text{ is true and so,}$$

$$\sqrt{\frac{2R}{r}}\cdot\left(2R+\frac{h_a+h_b+h_c}{2}-r\right) \geq 2(n_a+n_b+n_c)-\sum_{cyc}\sqrt{(b-c)^2+r^2} \to (m)$$

$$\text{and now, } max(r_a,r_b,r_c) \geq \sqrt{4R^2-7r^2}$$

$$\left(\begin{array}{c}\text{Reference : Inequality in Triangle by Dang Ngoc Minh} - 329;\\ \text{published at www.ssmrmh.ro}\end{array}\right) \overset{?}{\geq} 2R-r$$

$$\Leftrightarrow 4R^2-7r^2 \overset{?}{\geq} (2R-r)^2 = 4R^2-4Rr+r^2 \Leftrightarrow 4Rr \overset{?}{\geq} 8r^2 \to \text{true via Euler}$$

$$\therefore max(r_a,r_b,r_c) \geq 2R-r \to (n) \therefore (m) \text{ and } (n) \Rightarrow$$

$$\sqrt{\frac{2R}{r}}\cdot\left(max(r_a,r_b,r_c)+\frac{h_a+h_b+h_c}{2}\right) \geq 2(n_a+n_b+n_c)-\sum_{cyc}\sqrt{(b-c)^2+r^2}$$

$$\forall \text{ ABC}, ''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$